Linear Transformation

V, W vect spaces over $\mathbb{F}$

$$T : V \longrightarrow W$$

is called a linear transformation from V to W if,

i) $T(x+y) = T(x) + T(y)$
$\forall x, y \in V$

ii) $T(\alpha x) = \alpha T(x)$
$\forall \alpha \in \mathbb{F}$ & $\forall x \in V$

In Particular, if $V = W$

Then a l.t. from V to V

then it is called a linear operator on V

<u>Simple Property:</u>

$T: V \longrightarrow W$ l.t.

$$T(\alpha x) = \alpha\, T(x) \quad \forall\, \alpha \in F \ \& \ \forall\, x \in V$$

In particular if we take $\alpha = 0$, we get

$$T(0x) = 0\, T(x)$$

$$\Rightarrow T(\theta_V) = \theta_W$$

Thus any l.t. from V to W maps the θ_V to θ_W

Example 1

V n dimensional Vect space over $\mathbb{F}$

$\mathcal{B}$: $u_1, u_2, u_3, \ldots$ be an ordered basis for V

$x \in V \Rightarrow \quad x = x_1 u_1 + x_2 u_2 + \cdots + x_n u_n$

$$[x]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{F}^n$$

$$x \in V \xmapsto{\mathcal{B}} [x]_{\mathcal{B}}$$

We define

$$T_{\mathcal{B}} : V \longrightarrow \mathbb{F}^n$$

defined as

$$T_{\mathcal{B}}(x) = [x]_{\mathcal{B}}$$

We have already seen that this identification preserves $+$ and s.m

$\Longrightarrow$ $T_{\mathcal{B}}$ is a l.t. from V to $\mathbb{F}^n$

Thus every o.b. $\mathcal{B}$ for V induces a l.t. $T_{\mathcal{B}}$ from V to $\mathbb{F}^n$

EXAMPLE 2

$V = \mathbb{F}^n$; $W = \mathbb{F}^m$

Let A be any fixed $m \times n$ matrix in $\mathbb{F}^{m\times n}$

Define for every $x \in \mathbb{F}^n$,

$$T_A(x) = Ax \in \mathbb{F}^m \quad \forall x \in \mathbb{F}^n$$

Thus $T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$

Is T_A a l.t. from $\mathbb{F}^n$ to $\mathbb{F}^m$?

i) $x, y \in \mathbb{F}^n \Rightarrow T_A(x) = Ax$, $T_A(y) = Ay$

$$\Rightarrow T_A(x) + T_A(y) = Ax + Ay$$
$$= A(x+y)$$
$$= Az, \text{ when } z = x+y \in \mathbb{F}^n$$
$$= T_A(z)$$
$$= T_A(x+y)$$

$\Rightarrow T_A$ preserves addition

ii) $x \in V, \alpha \in \mathbb{F} \Rightarrow T_A(\alpha x) \overset{?}{=} \alpha T(x)$

$$x \in V, \alpha \in \mathbb{F} \Rightarrow T_A(x) = Ax$$

$$\Rightarrow \alpha T_A(x) = \alpha Ax$$
$$= A(\alpha x)$$
$$= A(z), \text{ when } z = \alpha x \in \mathbb{F}^n$$
$$= T_A(z)$$
$$= T_A(\alpha x)$$
$$\Rightarrow T_A(\alpha x) = \alpha T_A(x)$$

$\Rightarrow T_A$ preserves s.m

$\Rightarrow T_A$ is a l.t from $\mathbb{F}^n$ to $\mathbb{F}^m$

Thus every $m \times n$ matrix
$A \in \mathbb{F}^{m \times n}$ induces a l.t.
$$T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$$
by the def. $T_A(x) = Ax \; \forall x \in \mathbb{F}^n$

In particular, if we take $A \in \mathbb{F}^{n \times n}$
then A induces a l-t
$$T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^n$$
by the def
$$T_A(x) = Ax \; \forall x \in \mathbb{F}^n$$
i.e. T_A is a linear operator on $\mathbb{F}^n$

Thus every $n \times n$ matrix $A \in \mathbb{F}^{n \times n}$ induces a l.o on $\mathbb{F}^n$ as above

For example

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}_{2 \times 3}$$

induce a l.t. from $\mathbb{F}^3$ to $\mathbb{F}^2$

$$T_A : \mathbb{F}^3 \longrightarrow \mathbb{F}^2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto Ax$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + x_3 \\ x_2 + x_3 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \xmapsto{T_1} \begin{pmatrix} x_1 + x_3 \\ x_2 + x_3 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}_{2 \times 2}$$

$$T_A : F^2 \longrightarrow F^2$$

$$T_A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = Ax$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + x_2 \\ 3x_2 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xmapsto{T_A} \begin{pmatrix} x_1 + x_2 \\ 3x_2 \end{pmatrix}$$

EXAMPLE 3

Let $V = \mathbb{F}^{n \times n}$

Fix a $A \in \mathbb{F}^{n \times n}$

We define for every $X \in V = \mathbb{F}^{n \times n}$,

$$L_A(X) = AX \in \mathbb{F}^{n \times n} \quad \forall X \in \mathbb{F}^{n \times n}$$

Thus

$$L_A : \mathbb{F}^{n \times n} \longrightarrow \mathbb{F}^{n \times n}$$

Is L_A a l.o. on $\mathbb{F}^{n \times n}$?

i) $X, Y \in \mathbb{F}^{n\times n} \implies \left.\begin{array}{l} L_A(X) = AX \\ L_A(Y) = AY \end{array}\right\}$

$$\implies L_A(X) + L_A(Y) = AX + AY$$
$$= A(X+Y)$$
$$= L_A(X+Y)$$

$\implies L_A$ preserves addition

ii) $\alpha \in \mathbb{F}, X \in \mathbb{F}^{n\times n} \implies L_A(X) = AX$

$$\implies \alpha L_A(x) = \alpha AX$$
$$= A(\alpha X)$$
$$= L_A(\alpha X)$$

$\implies L_A$ preserves s m

$\Rightarrow L_A$ is a l.o. on $\mathbb{F}^{n\times n}$

Thus every fixed $A \in \mathbb{F}^{n\times n}$ induces a l.o. $\hat{L_A}$ on $\mathbb{F}^{n\times n}$ as above by left multiplication by A

Similarly given $A \in \mathbb{F}^{n\times n}$, it induces a l.o. on $\mathbb{F}^{n\times n}$ — denoted by R_A — by right multiplication by A

$$R_A(X) = XA \quad \forall X \in \mathbb{F}^{n\times n}$$

EXAMPLE

Let $V = \mathbb{F}^{n \times n}$

Let A, B be fixed matrices in $\mathbb{F}^{n \times n}$. For any $X \in \mathbb{F}^{n \times n}$ define

$$T_{A,B}(X) = AXB \in \mathbb{F}^{n \times n} \quad \forall X \in \mathbb{F}^{n \times n}$$

Hence

$$T_{A,B} : \mathbb{F}^{n \times n} \longrightarrow \mathbb{F}^{n \times n}$$

It is easy to verify that $T_{A,B}$

is a l.o. on $\mathbb{F}^{n\times n}$.

In particular let P be a fixed $n\times n$ matrix in $\mathbb{F}^{n\times n}$ s.t. P^{-1} exists

Take $A = P^{-1}$, $B = P$

We can define

$$T_{P^{-1},P} = \text{denote this as } T_P$$

$$T_P : \mathbb{F}^{n\times n} \longrightarrow \mathbb{F}^{n\times n}$$

$$T_P(X) = P^{-1}XP \quad \forall X \in \mathbb{F}^{n\times n}$$

T_P is a l.o on $\mathbb{F}^{n\times n}$

Def: Let $X \in \mathbb{F}^{n\times n}$

We say a $Y \in \mathbb{F}^{n\times n}$ is SIMILAR to X if

$\exists P \in \mathbb{F}^{n\times n}$ s.t

$$T_P(X) = Y$$

We call such l.o. T_P on $\mathbb{F}^{n\times n}$

as similarity transformations

EXAMPLE

$V = \mathbb{F}^{m \times n}$

Let Q be a fixed $m \times m$ matrix in $\mathbb{F}^{m \times m}$

For $X \in \mathbb{F}^{m \times n}$ define

$$L_Q(X) = \underbrace{\underset{m \times m}{Q}\ \underset{m \times n}{X}}_{m \times n} \in \mathbb{F}^{m \times n} \quad \forall X \in \mathbb{F}^{m \times n}$$

$$\Rightarrow \quad L_Q : \mathbb{F}^{m \times n} \longrightarrow \mathbb{F}^{m \times n}$$

Is L_Q a l.o. on $\mathbb{F}^{m \times n}$?

It is easy (as in the case of square matrices) to verify that

L_Q preserves $+$ and s.m

$\Rightarrow L_Q$ is a l.o. on $\mathbb{F}^{m\times n}$

Any $m \times m$ matrix $Q \in \mathbb{F}^{m\times m}$ induces a l.o. on $\mathbb{F}^{m\times n}$ by premultiplication by Q

Similarly every $n \times n$ matrix $P \in \mathbb{F}^{n\times n}$ induces a l.o. R_P

on $\mathbb{F}^{m\times n}$, as Post multiplication by P, i.e.

$$R_P(X) = XP \quad \forall\, X \in \mathbb{F}^{m\times n}$$

$$\underbrace{\underset{m\times n}{X}\ \underset{n\times n}{P}}_{m\times n} \in \mathbb{F}^{m\times n}$$

Fix any $Q \in \mathbb{F}^{m\times m}$, $P \in \mathbb{F}^{n\times n}$

Then define

$$T_{Q,P}(X) = QXP \quad \forall\, X \in \mathbb{F}^{m\times n}$$

$$\underbrace{\underset{m\times m}{Q}\ \underset{m\times n}{X}\ \underset{n\times n}{P}}_{m\times n} \in \mathbb{F}^{m\times n} \ \forall\, X \in \mathbb{F}^{m\times n}$$

Hence

$$T_{Q,P} : \mathbb{F}^{m \times n} \longrightarrow \mathbb{F}^{m \times n}$$

$(T_{Q,P})$ is a l.o on $\mathbb{F}^{m \times n}$

EXAMPLE

$V = \mathbb{F}[x]$

For any $p \in \mathbb{F}[x]$, define

$$D(p) = \frac{dp}{dx} \in \mathbb{F}[x] \quad \forall p \in \mathbb{F}[x]$$

$$\Rightarrow \qquad D : \mathbb{F}[x] \longrightarrow \mathbb{F}[x]$$

Is D a l.o. on $\mathbb{F}[x]$?

i) $p, q \in \mathbb{F}[x] \Longrightarrow D(p) = \frac{dp}{dx}, D(q) = \frac{dq}{dx}$

$$\Longrightarrow D(p) + D(q) = \frac{dp}{dx} + \frac{dq}{dx}$$

$$= \frac{d}{dx}(p+q)$$

$$= D(p+q)$$

$\Longrightarrow$ D preserves addition

ii) $\alpha \in \mathbb{F}, p \in \mathbb{F}[x] \Longrightarrow D(\alpha p) = \frac{d}{dx}(\alpha p(x))$

$$= \alpha \frac{d}{dx} p$$

$$= \alpha D(p)$$

$\Longrightarrow$ D preserves s.m

$\Rightarrow$ D is a l.o. on $\mathbb{F}[x]$

EXAMPLE:

$V = \mathbb{F}_4[x]$, $W = \mathbb{F}_3[x]$

$T : V \longrightarrow W$

as $T(p) = \dfrac{d^2}{dx^2}(p) \in W \quad \forall\, p \in \mathbb{F}_4[x]$

It is easy to see that

$$T(p+q) = T(p) + T(q) \quad \forall\, p, q \in \mathbb{F}_4[x]$$

$$T(\alpha p) = \alpha\, T(p) \quad \forall\, \alpha \in \mathbb{F} \quad \forall\, p \in \mathbb{F}_4[x]$$

$\Rightarrow$ T is l.t from $\mathbb{F}_4[x]$ to $\mathbb{F}_3[x]$