$\left.\begin{array}{l} V \text{ vector space} \\ W \quad\quad\quad\quad " \end{array}\right\}$ over $\mathbb{F}$

Recall

$$T : V \longrightarrow W$$

is called a l.t. if

$$T(x+y) = T(x) + T(y) \ \forall\, x, y \in V,$$

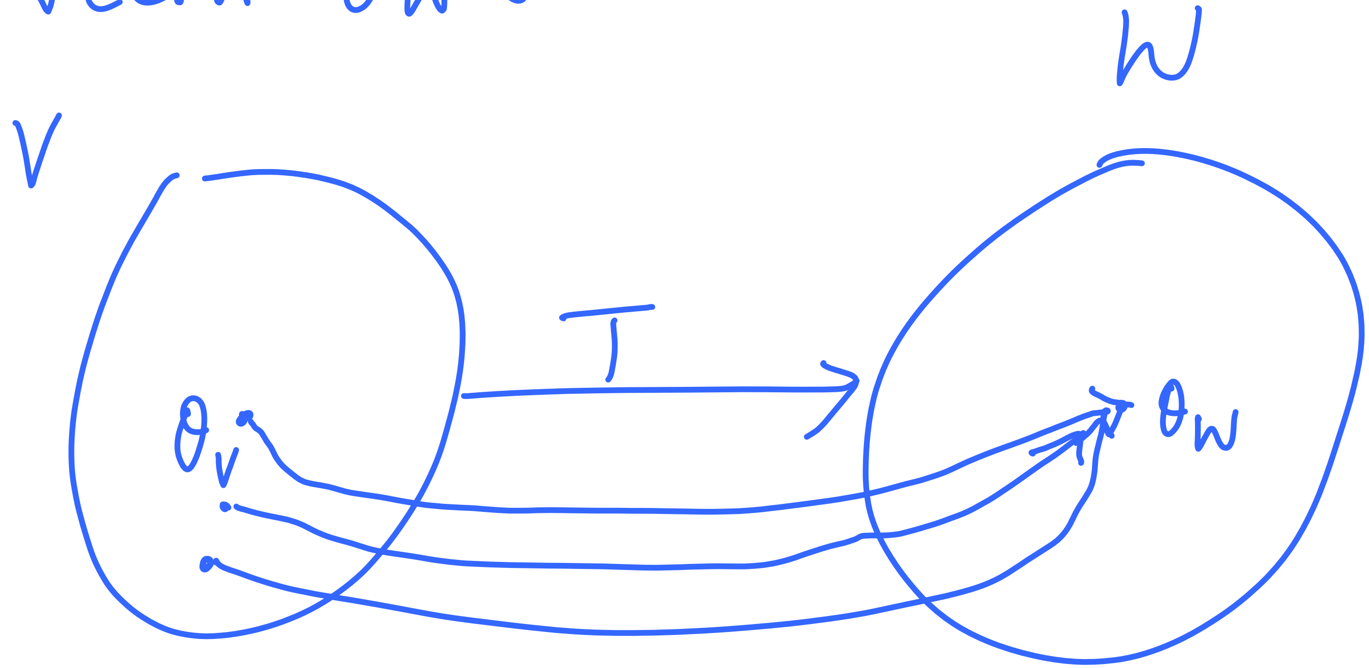
and

$$T(\alpha x) = \alpha T(x) \ \forall\, \alpha \in \mathbb{F} \ \& \ \forall\, x \in V$$

<u>A simple property of a l.t</u>

$$T(\theta_V) = \theta_W$$

A l.t. always maps the zero vector $\theta_V \in V$ to the zero vector $\theta_W \in W$



There may be several vectors in V that get mapped to θ_W

We collect all these vectors in V

which get mapped to θ_W

$$N_T = \{ x \in V : Tx = \theta_W \}$$

Clearly 1) N_T is a subset of V, and

2) $\theta_V \in N_T$ since $T(\theta_V) = \theta_W$

$\Rightarrow$ N_T is a NONEMPTY subset of V

Is N_T a subspace of V?

For this to happen N_T must be closed w.r.t addition & scalar multiplication

1) $x, y \in N_T \implies Tx = \theta_W \, , \; Ty = \theta_W$

$\implies Tx + Ty = \theta_W$

$\implies T(x+y) = \theta_W$ (since T is a l.T)

$\implies x + y \in N_T$

$\implies$ N_T is closed w.r.t. addition

2) $x \in N_T \implies Tx = \theta_W$

$\implies \alpha Tx = \alpha \theta_W = \theta_W \quad \forall \alpha \in F$

$\implies T(\alpha x) = \theta_W$ (since T is a l.t.)

$\implies \alpha x \in N_T$

$\implies$ N_T is closed under scalar mult.

Hence

N_T is a Nonempty subset of V which is closed under addition and scalar mult

$\Longrightarrow$ N_T is a subspace of V

This subspace is called the NULL SPACE of T

$$N_T, \text{ (Null Space of } T) = \{x \in V : Tx = \theta_W\}$$

is a subspace of V.

Remark:

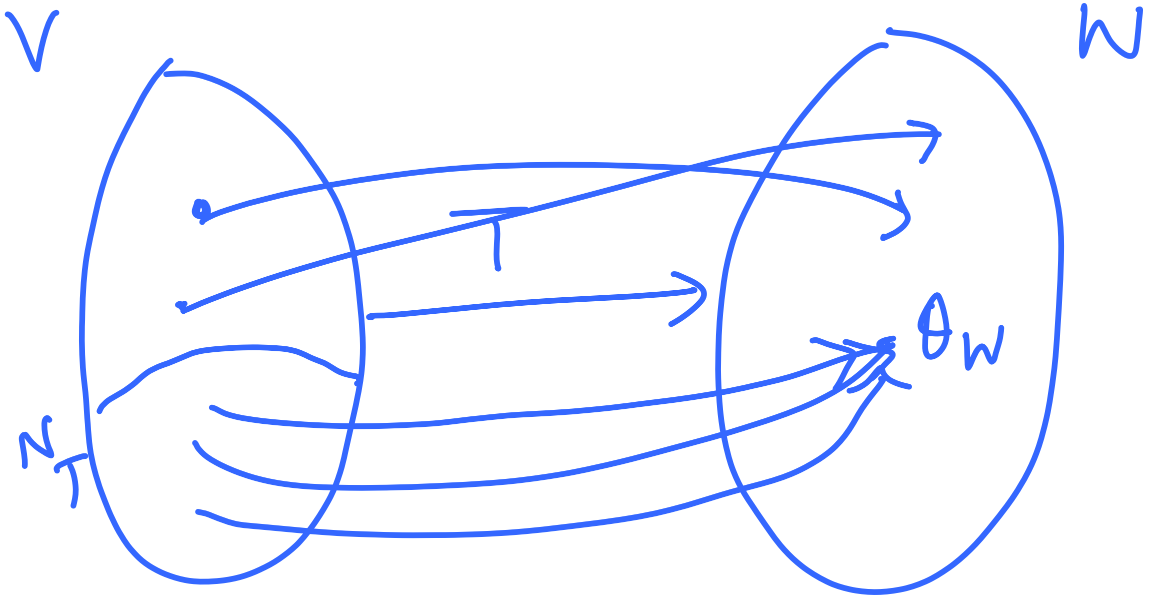
Suppose V is a f.d.v.s, W v.s & $T: V \longrightarrow W$ l.t.

N_T is a subspace of V

$\Rightarrow$ N_T is finite dimensional and $\dim N_T \leq \dim V$

Thus $\dim N_T$ Nullity of T and is denoted by ν_T

$$\boxed{\nu_T = \dim N_T}$$



$$R_T = \{ y \in W : \exists\, x \in V \ni Tx = y \}$$

Clearly i) R_T is a subset of W, and

ii) $\theta_W \in R_T$ since $\theta_V \in V$ and $T(\theta_V) = \theta_W$

$\Rightarrow$ R_T is a Nonempty Subset of W

Is R_T a subspace of W?

1) $y_1, y_2 \in R_T \Rightarrow \exists\, x_1 \in V$ s.t $T(x_1) = y_1$,

$\exists\, x_2 \in V$ s.t $T(x_2) = y_2$

$\Rightarrow \exists\, x_1, x_2 \in V$ s.t $T(x_1) + T(x_2) = y_1 + y_2$

$\Rightarrow \exists\, x_1, x_2 \in V$ s.t. $T(x_1 + x_2) = y_1 + y_2$

(since T is l.t.)

$\Rightarrow \exists\, z = x_1 + x_2 \in V$ s.t. $T(z) = y_1 + y_2$

$\Rightarrow y_1 + y_2 \in R_T$

$\Rightarrow$ R_T is closed under addition

2) $y \in R_T \Rightarrow \exists x \in V$ s.t. $Tx = y$

$\Rightarrow \exists x \in V$ s.t. $\alpha Tx = \alpha y \ \forall \alpha \in \mathbb{F}$

$\Rightarrow \exists x \in V$ s.t. $T(\alpha x) = \alpha y$

$\Rightarrow \exists z = \alpha x \in V$ s.t $T(z) = \alpha y$

$\Rightarrow \alpha y \in R_T$

$\Rightarrow R_T$ is closed under scalar mult

Hence

R_T is a Nonempty subset of W which is closed under addition & scalar mult

$\Rightarrow R_T$ is a subspace of W

This subspace is called RANGE of T

(Range T), $R_T = \{y \in W : \exists\, x \in V \ni T(x) = y\}$

is a subspace of W

If W is a fdvs then

R_T will also be f-dimensional

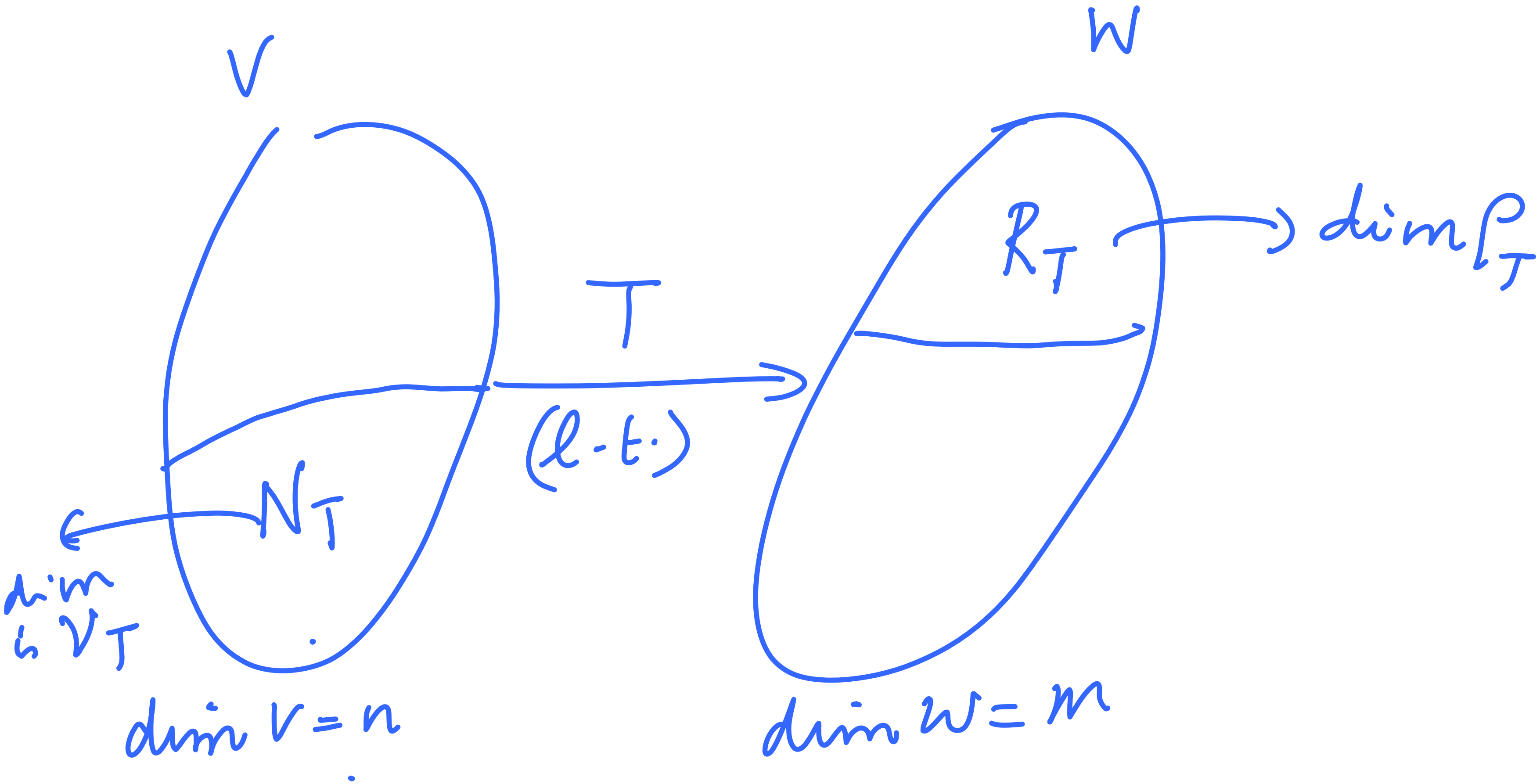
$\dim R_T \leq \dim W$

This dim is called RANK of T

and is denoted by ρ_T

$$\boxed{\rho_T = \dim \text{ of } R_T}$$

$$\boxed{\rho_T \le \dim W}$$



$\underline{\underline{\nu_T \leq n}}$ $\underline{\underline{\rho_T \leq m}}$

EXAMPLES

(1) $V = \mathbb{F}^n$, $W = \mathbb{F}^m$

Let $A \in \mathbb{F}^{m \times n}$ a fixed matrix

Define

$$T_A : \mathbb{F}^n \longrightarrow \mathbb{F}^m$$

as $T_A(x) = Ax$

We have seen that T_A is a l-t

Null space of T_A

$$x \in N_{T_A} \iff T_A(x) = \theta_W$$
$$\iff Ax = \theta_m$$
$$\iff x \in N_A$$

(i.e. the set of all sol of the HS $Ax = \theta_m$)

For example $m=2$, $n=3$,

$$V = \mathbb{F}^3, \quad W = \mathbb{F}^2$$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$T_A(x) = Ax$$

$$N_{T_A} = N_A = \{x \in \mathbb{F}^n : Ax = \theta_m\}$$

$$A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 = x_3, \quad x_2 = x_3$$

$$N_A = \left\{ x = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{F} \right\}$$

We have : $N_{T_A} = \hat{}$

Since $u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a basis for N_{T_A}

$\dim N_{T_A} = 1$

Nullity of T_A, $\nu_T = 1$

Range T_A. $V = \mathbb{F}^n$, $W = \mathbb{F}^m$,

$A \in \mathbb{F}^{m \times n}$

$$T_A(x) = Ax$$

$$R_{T_A} = \{ y \in \mathbb{F}^m : \exists\, x \in \mathbb{F}^n \ni T_A(x) = y \}$$

$$= \{ y \in \mathbb{F}^m : \exists\, x \in \mathbb{F}^n \ni Ax = y \}$$

$$= \{ y \in \mathbb{F}^m : \text{The NHS } Ax = y \text{ has a sol} \}$$

$=$ Range of the Matrix A

$$\dim R_{T_A} = \dim \text{Range } A$$
$$\text{Rank } T_A = \text{Rank of } A$$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} :$$

$$T_A : \mathbb{F}^3 \longrightarrow \mathbb{F}^2$$
$$T_A(x) = Ax$$

<u>NHS</u> $\quad Ax = y$

$$\begin{pmatrix} x_1 - x_3 \\ x_2 - x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Whatever $y_1, y_2 \in \mathbb{F}$ the system has a sol $x_1 = y_1,\ x_2 = y_2,\ x_3 = 0$

$$\therefore \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$\Rightarrow$ Every $y \in F^2$ belongs to R_{T_A}

$\Rightarrow R_{T_A} = F^2$

Hence $\text{Rank } T_A = \rho_{T_A} = 2$

Example 2

$V = \mathbb{F}_4[x]$

$D : V \longrightarrow V$

defined as

$$D(p) = \frac{dp}{dx}$$

We have seen D is a l.o. on $\mathbb{F}_4[x]$

Null space of D

$p \in N_D \iff Dp = 0$

$\iff \frac{dp}{dx} = 0$

$\Longleftrightarrow \quad p = \text{const}$

$$\Longrightarrow \quad N_D = \{ p \in \mathbb{F}_4[x] : p = a_0 ,\ a_0 \in \mathbb{F} \}$$

$p = 1$ is a basis for N_D

$\therefore \dim N_D = 1$

$\nu_D = 1$

Range of D

To find all poly $p \in \mathbb{F}_4[x]$
for which we can find

a $q \in \mathbb{F}_4[x]$ s.t

$$D(q) = p$$

$$\frac{dq}{dx} = p$$

$$q(x) = \int_0^x p(x)\,dx$$

In order that $q \in \mathbb{F}_4[x]$ it is necessary that $p \in \mathbb{F}_3[x]$

Hence $R_D = \left\{ p \in \mathbb{F}_4(x) : p \in \mathbb{F}_3[x] \quad p = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \quad a_j \in \mathbb{F} \right\}$

$\dim R_D = 4$

Rank D, $\rho_D = 4$

EXAMPLE 3

$V = \mathbb{F}_4[x]$, $W = \mathbb{F}_3[x]$

$T : V \longrightarrow W$ defined

as $T(p) = \dfrac{d^2 p}{dx^2}$

T is a l.t from V to W

Null space of T

$$p \in N_T \iff Tp = 0$$

$$\iff \frac{d^2p}{dx^2} = 0$$

$$\iff p(x) = a_0 + a_1 x \,,\ a_0, a_1 \in \mathbb{F}$$

$$N_T = \left\{ p \in \mathbb{F}_4[x] : p(x) = a_0 + a_1 x \,;\ a_0, a_1 \in \mathbb{F} \right\}$$

$p_1 = 1$, $p_2 = x$ is a basis for N_T

$$\dim N_T = 2$$

$$\implies \nu_T = 2$$

Range of T

To find all poly $p \in \mathbb{F}_3[x]$ for which $\exists\ q \in \mathbb{F}_4[x]$ s.t

$$T(q) = p$$

$$\frac{d^2 q}{dx^2} = p$$

Since $q \in \mathbb{F}_4[x]$ we have $\frac{d^2 q}{dx^2} \in \mathbb{F}_2[x]$

& hence p has to be in $\mathbb{F}_2[x]$

For every $p \in \mathbb{F}_2[x]$ if we define

$$q = \int_0^x \left(\int_0^x p(x)\, dx \right) dx$$

then $q \in \mathbb{F}_4[x]$ and

$$Dq = \frac{d^2 q}{dx^2} = p$$

& hence $p \in R_D$

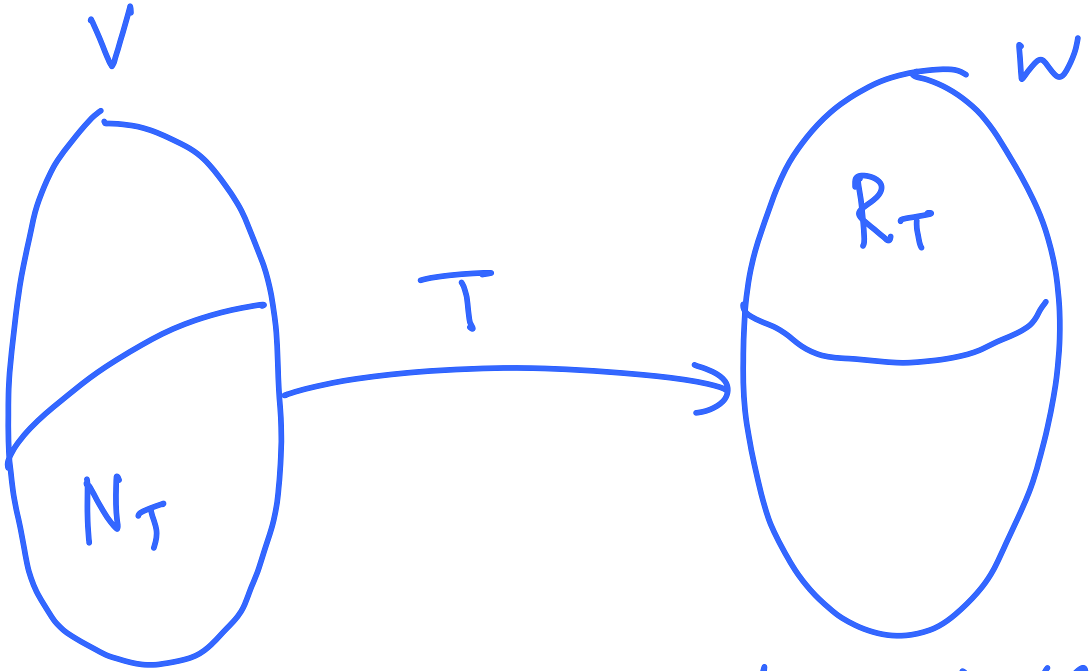
Therefore $R_D = \mathbb{F}_2[x]$

$$R_D = \mathbb{F}_2[x]$$

$$\dim R_D = 3$$

$$\rho_D = 3$$

$$\text{Rank } D + \text{Nullity } D = 2+3 = 5$$
$$= \dim V$$



$\dim V = n$

$\dim N_T = \nu_T$

$\nu_T \leq n$

$\dim W = m$

$\dim R_T = \rho_T$

$\rho_T \leq m$