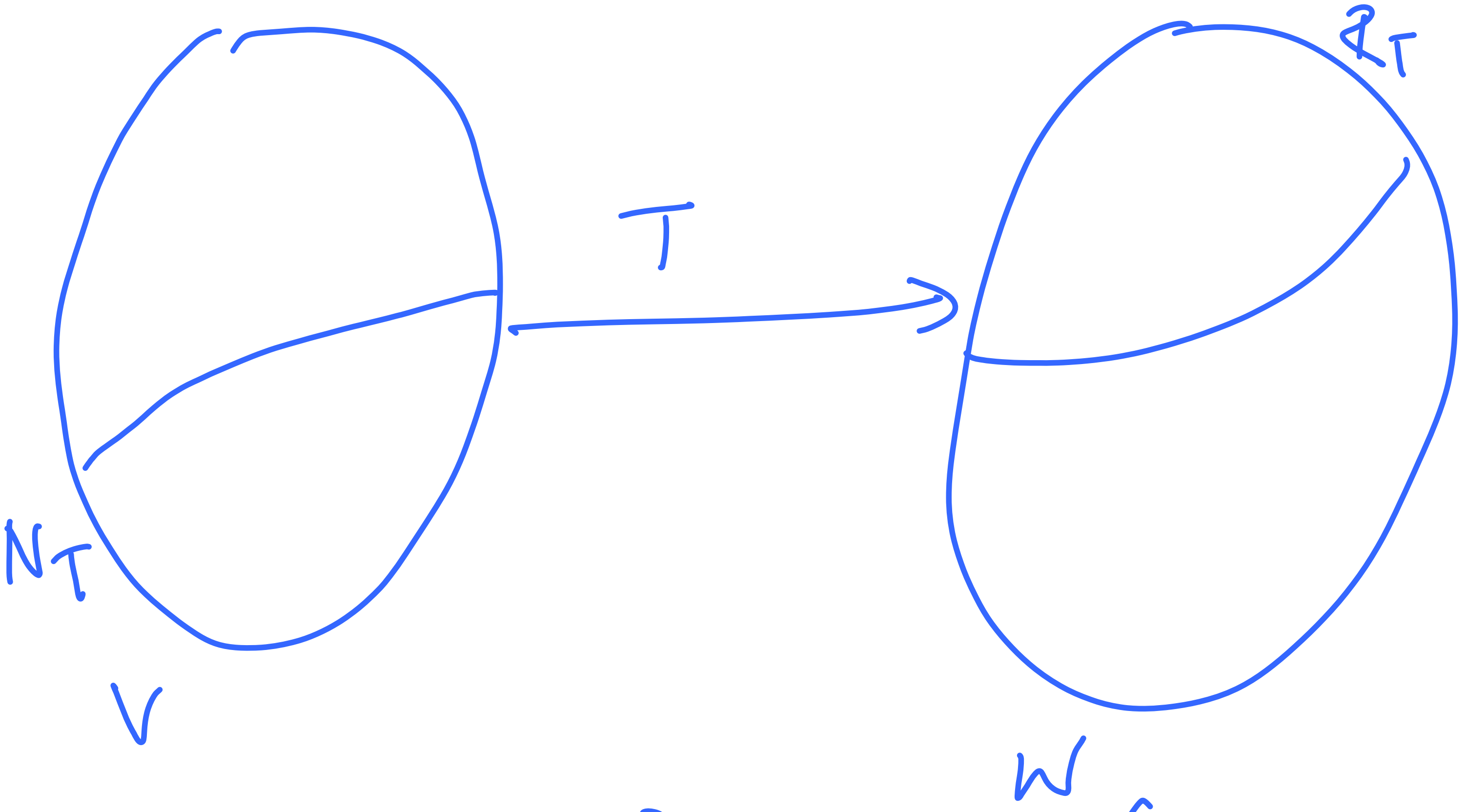
Range, Null Space



$$N_T = \{x \in V : T(x) = \theta_W\}$$

Subspace of V

$$R_T = \{y \in W : \exists\, x \in V \ni Tx = y\}$$

Subspace of W

dim N_T, ν_T : Nullity of T

dim R_T, ρ_T
Rank of T

$\dim N_T = \nu_T \le \dim V$

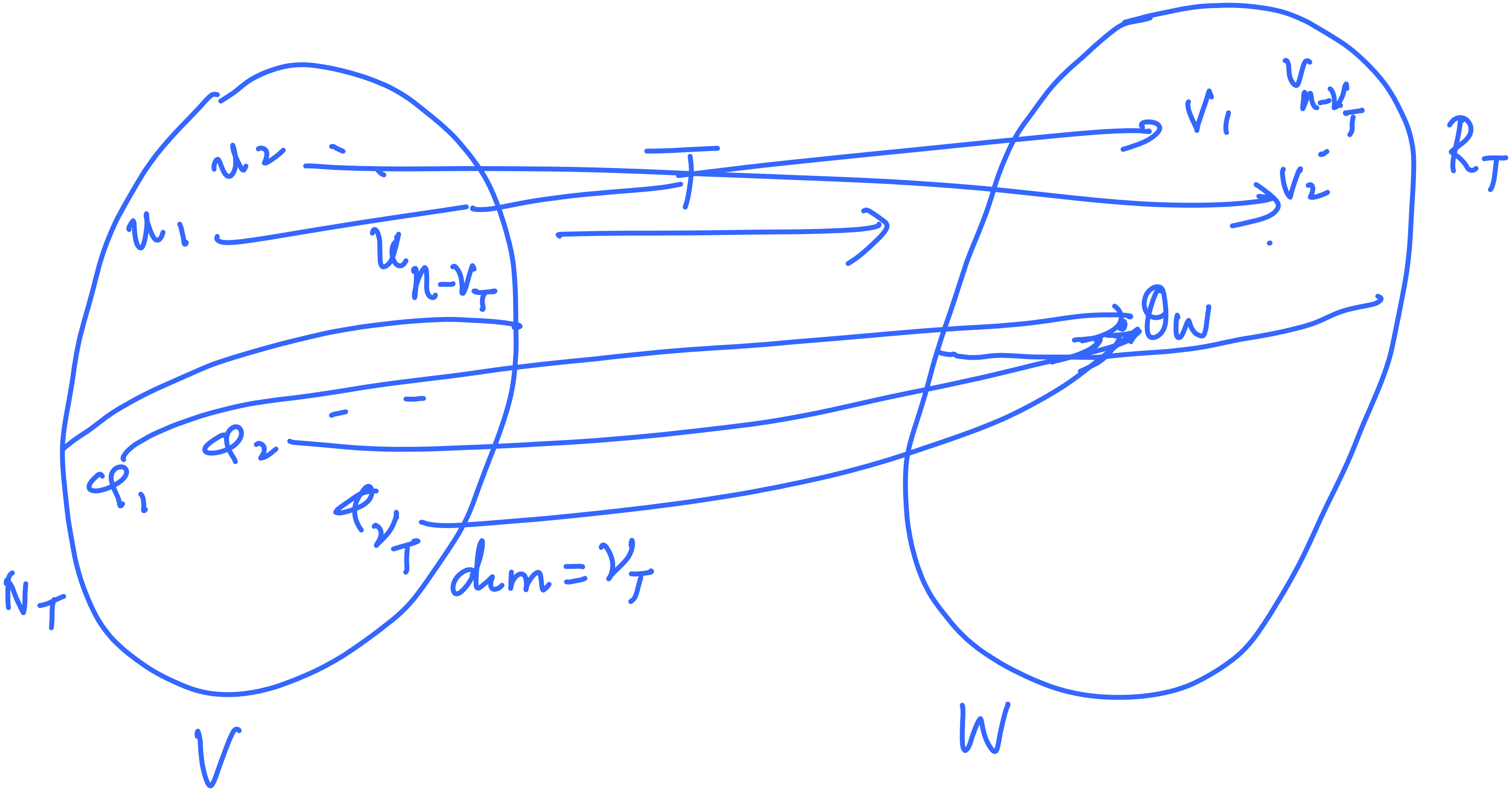
$\dim R_T = \rho_T \le \dim W$

Let
$\dim V = n$

$\dim W = m$

$\nu_T \le n$

$\rho_T \le m$



Any basis for N_T must have ν_T vectors. Let

$$B_N = \varphi_1, \varphi_2, \ldots, \varphi_{\nu_T}$$

be a basis for N_T

We can extend B_N to a basis

$$B_V = \varphi_1, \ldots, \varphi_{\nu_T}, u_1, u_2, \ldots, u_{n-\nu_T}$$

for V by appending $n-\nu_T$ l.i. vectors from $V \setminus N_T$

We have

$$T(\varphi_1) = T(\varphi_2) = \cdots = T(\varphi_{\nu_T}) = \theta_W$$

$$T(u_1),\ T(u_2),\ \cdots,\ T(u_{n-\nu_T}) \neq \theta_W$$

$$x \in V \Rightarrow x = \alpha_1 \varphi_1 + \alpha_2 \varphi_2 + \cdots + \alpha_{\nu_T} \varphi_{\nu_T} + \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_{n-\nu_T} u_{n-\nu_T}$$

$$\Rightarrow Tx = T(\alpha_1 \varphi_1) + \cdots + T(\alpha_{\nu_T} \varphi_{\nu_T}) + T(\beta_1 u_1) + \cdots + T(\beta_{n-\nu_T} u_{n-\nu_T})$$

$$\Rightarrow Tx = \alpha_1 T\varphi_1 + \cdots + \alpha_{\nu_T} T\varphi_{\nu_T} + \beta_1 Tu_1 + \cdots + \beta_{n-\nu_T} Tu_{n-\nu_T}$$

$$\Rightarrow Tx = \beta_1 Tu_1 + \cdots + \beta_{n-\nu_T} Tu_{n-\nu_T}$$

$$\Rightarrow Tx = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{n-\nu_T} v_{n-\nu_T}$$

where $v_1 = Tu_1, v_2 = Tu_2, \ldots, v_{n-\nu_T} = Tu_{n-\nu_T}$

But every vector in R_T is of the form Tx for some $x \in V$

Hence every vect in R_T is of the form

$$\left\{ \begin{array}{l} \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_{n-\nu_T} v_{n-\nu_T} \\ \& \; v_1, v_2, \ldots, v_{n-\nu_T} \in R_T \end{array} \right.$$

$\Longrightarrow \quad S = v_1, v_2, \ldots, v_{n-\nu_T}$

is a spanning set for R_T

If $\varphi_1, \ldots, \varphi_{\nu_T}$ is a basis for N_T

& $\varphi_1, \ldots, \varphi_{\nu_T}, u_1, \ldots, u_{n-\nu_T}$ is an extension to a basis for V

then T of the extending vector form a spanning set for R_T

Natural to ask if S is a basis for R_T.

S will be a basis for R_T

if it is l.i

S will be l.i. if

$$a_1 v_1 + a_2 v_2 + \cdots + a_{n-\nu_T} v_{n-\nu_T} = \theta_W$$?

$$\Longrightarrow a_1 = a_2 = \cdots = a_{n-\nu_T} = 0$$

$$a_1 v_1 + \cdots + a_{n-\nu_T} v_{n-\nu_T} = \theta_W$$

$$\Longrightarrow \sum_{j=1}^{n-\nu_T} a_j v_j = \theta_W$$

$$\Longrightarrow \sum_{j=1}^{n-\nu_T} a_j T(u_j) = \theta_W$$

$$\Rightarrow \sum_{j=1}^{n-\nu_T} T(a_j u_j) = \theta_W$$

$$\Rightarrow T\left(\sum_{j=1}^{n-\nu_T} a_j u_j\right) = \theta_W$$

$$\Rightarrow T(x) = \theta_W \text{ where } x = \sum_{j=1}^{n-\nu_T} a_j u_j \in V$$

$$\Rightarrow x \in N_T$$

$$\Rightarrow x = \sum_{i=1}^{\nu_T} b_i \varphi_i$$

$$\Rightarrow \sum_{j=1}^{n-\nu_T} a_j u_j = \sum_{i=1}^{\nu_T} b_i \varphi_i$$

$$\Rightarrow$$

$$\sum_{j=1}^{n-\nu_T} a_j u_j + \sum_{i=1}^{\nu_T} (-b_i)\varphi_i = \theta_V$$

$$\Longrightarrow \quad a_j = 0 \quad 1 \le j \le n-\nu_T$$

$$(\; b_i = 0 \quad 1 \le i \le \nu_T)$$

Hence $\displaystyle\sum_{j=1}^{n-\nu_T} a_j v_j = \theta_W \Rightarrow a_j = 0 \quad 1 \le j \le n-\nu_T$

$\Rightarrow$

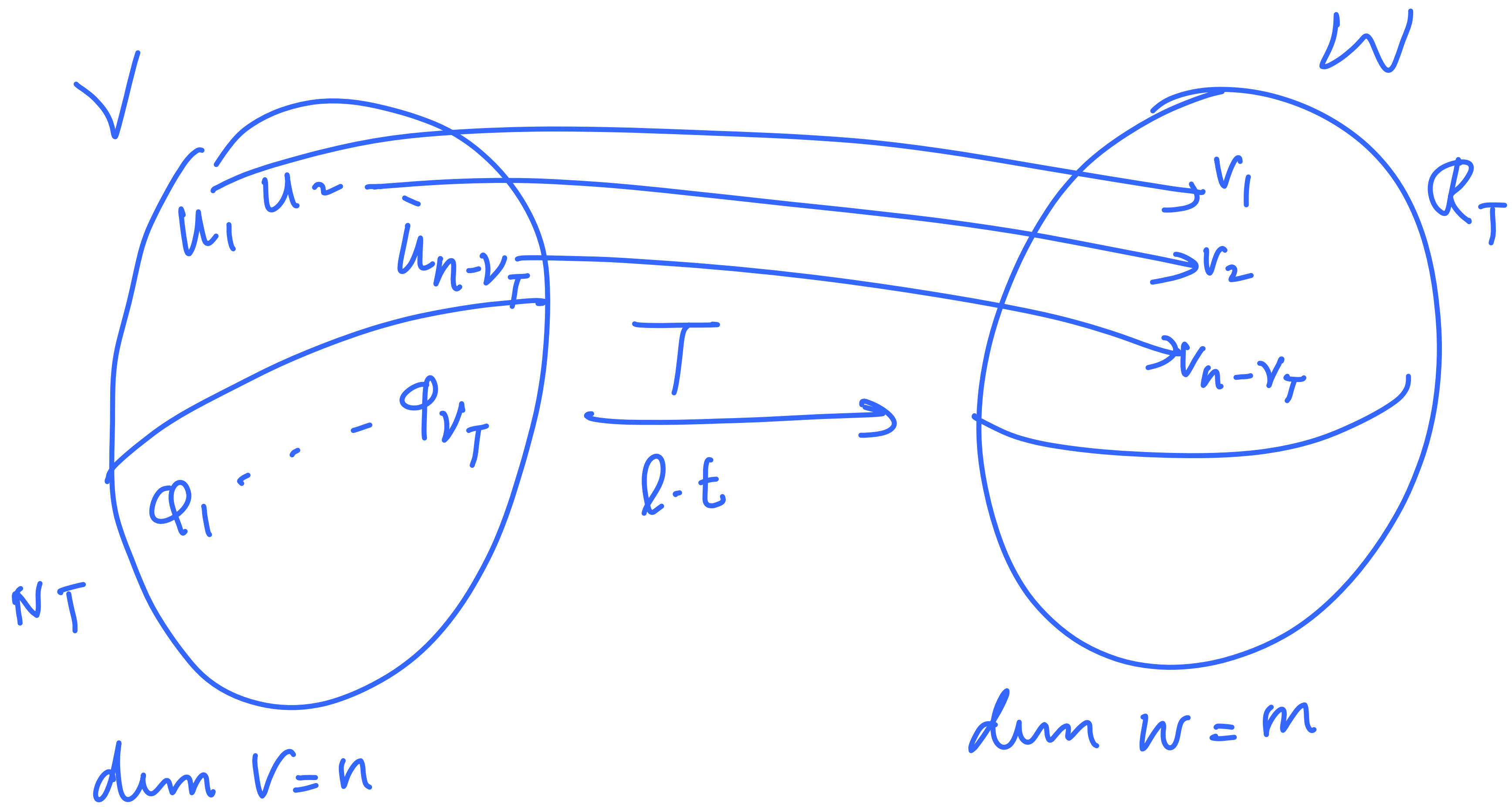
$v_1, v_2, \dots, v_{n-\nu_T}$ l-i

Hence

$S = v_1, v_2, \ldots, v_{n-\nu_T}$

is a l.i. spanning set for R_T

& hence S is a basis for R_T



Since $v_1, v_2, \ldots, v_{n-\nu_T}$ is a basis for R_T & this has $n-\nu_T$ vectors we get:

$$\dim R_T = n - \nu_T$$

$$\implies \rho_T = n - \nu_T$$

$$\implies \rho_T + \nu_T = n$$

$$\implies \text{Rank } T + \text{Nullity } T = \dim V$$

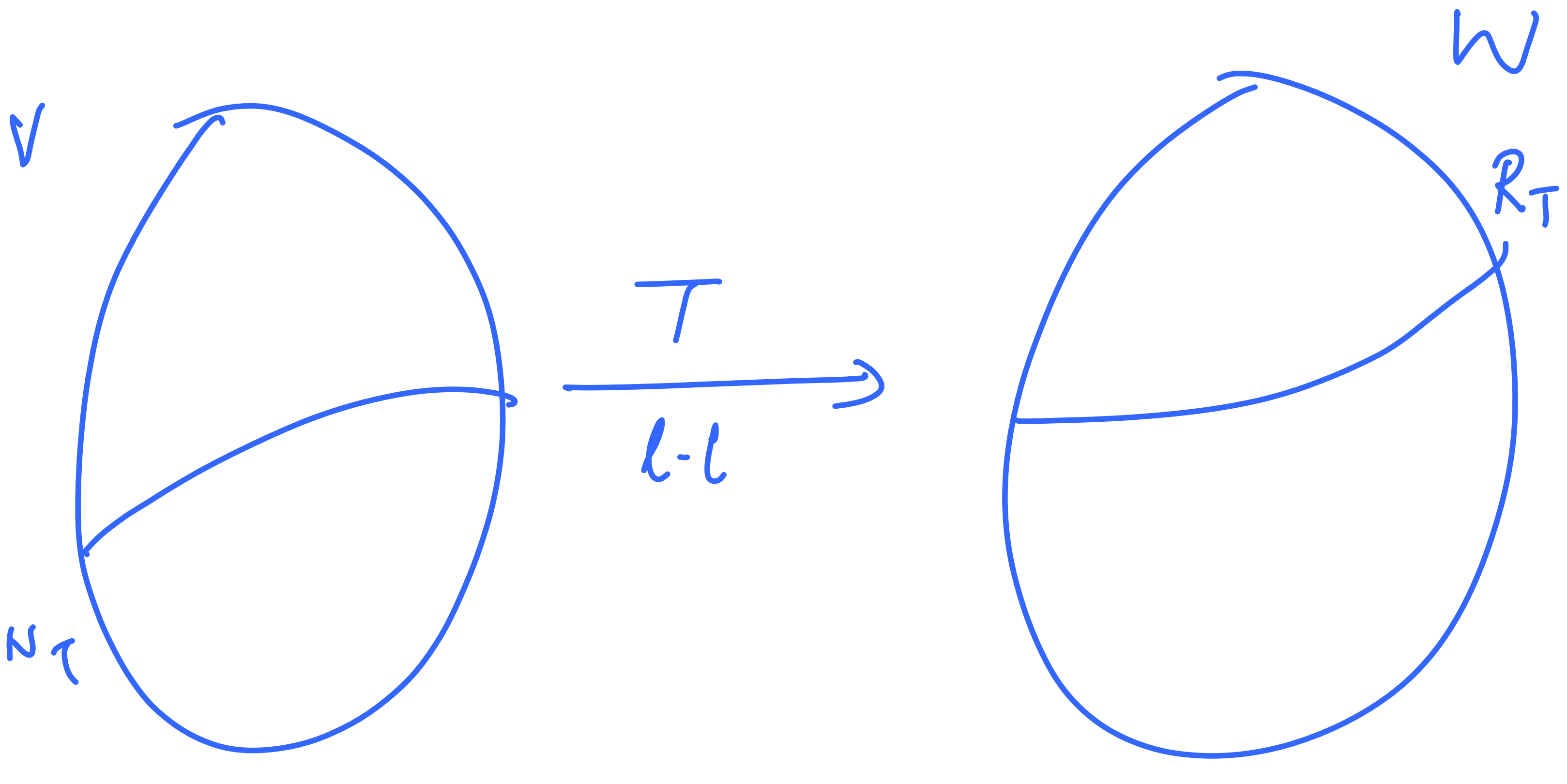
We have

RANK - NULLITY THEOREM

V, W f.d.v.s over F

$T : V \longrightarrow W$ l.t.

Then Rank T + Nullity T = dim V



$\nu_T = \dim N_T \leq \dim V$

$\rho_T = \dim R_T \leq \dim W$

$\rho_T \leq \dim V$

(by Rank Nullity Theorem

Rank $T \leq \dim V$, as well as, $\dim W$

EXAMPLES

(1) $V = \mathbb{F}^3 \qquad W = \mathbb{F}^2$

$T_A : \mathbb{F}^3 \longrightarrow \mathbb{F}^2$

defined as

$T_A(x) = Ax$ where

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

We had found

$$N_T = \left\{ x \in V ; x = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{F} \right\}$$

$$\nu_T = \dim N_T = 1$$

We had found

$$R_T = \mathbb{F}^2$$

$$\rho_T = \dim R_T = \dim \mathbb{F}^2 = 2$$

$\rho_T + \nu_T = 2 + 1 = 3 = \dim V \;(= \dim \mathbb{F}^3$

$\operatorname{rank} T + \operatorname{nullity} T = \dim V$

<u>Ex 2</u> $\quad V = \mathbb{F}_4[x]$

$$D : V \longrightarrow V$$

$$D(p) = \frac{dp}{dx}$$

We had

$$N_T = \{ p \in \mathbb{F}_4[x] : p(x) = a_0 ,\ a_0 \in \mathbb{F} \}$$

$\nu_T = \dim N_T = 1$

We also had

$$R_T = \{ p \in \mathbb{F}_3[x] \}$$

$$\rho_T = \dim R_T = \dim \mathbb{F}_3 = 4$$

$$\rho_T + \nu_T = 4 + 1 = 5 = \dim \mathbb{F}^4$$
$$= \dim V$$

<u>Ex 3</u> $V = \mathbb{F}_4[x]$; $W = \mathbb{F}_3[x]$

$$T : V \longrightarrow W$$

defined as
$$T(p) = \frac{d^2 p}{dx^2}$$

We had

$$N_T = \{ p \in \mathbb{F}_4[x] : p(x) = a_0 + a_1 x ; a_0, a_1 \in \mathbb{F} \}$$

$$\nu_T = \dim N_T = 2$$

We also had

$$R_T = \{ p \in \mathbb{F}_2[x] \}$$

$$\rho_T = \dim R_T = 3$$

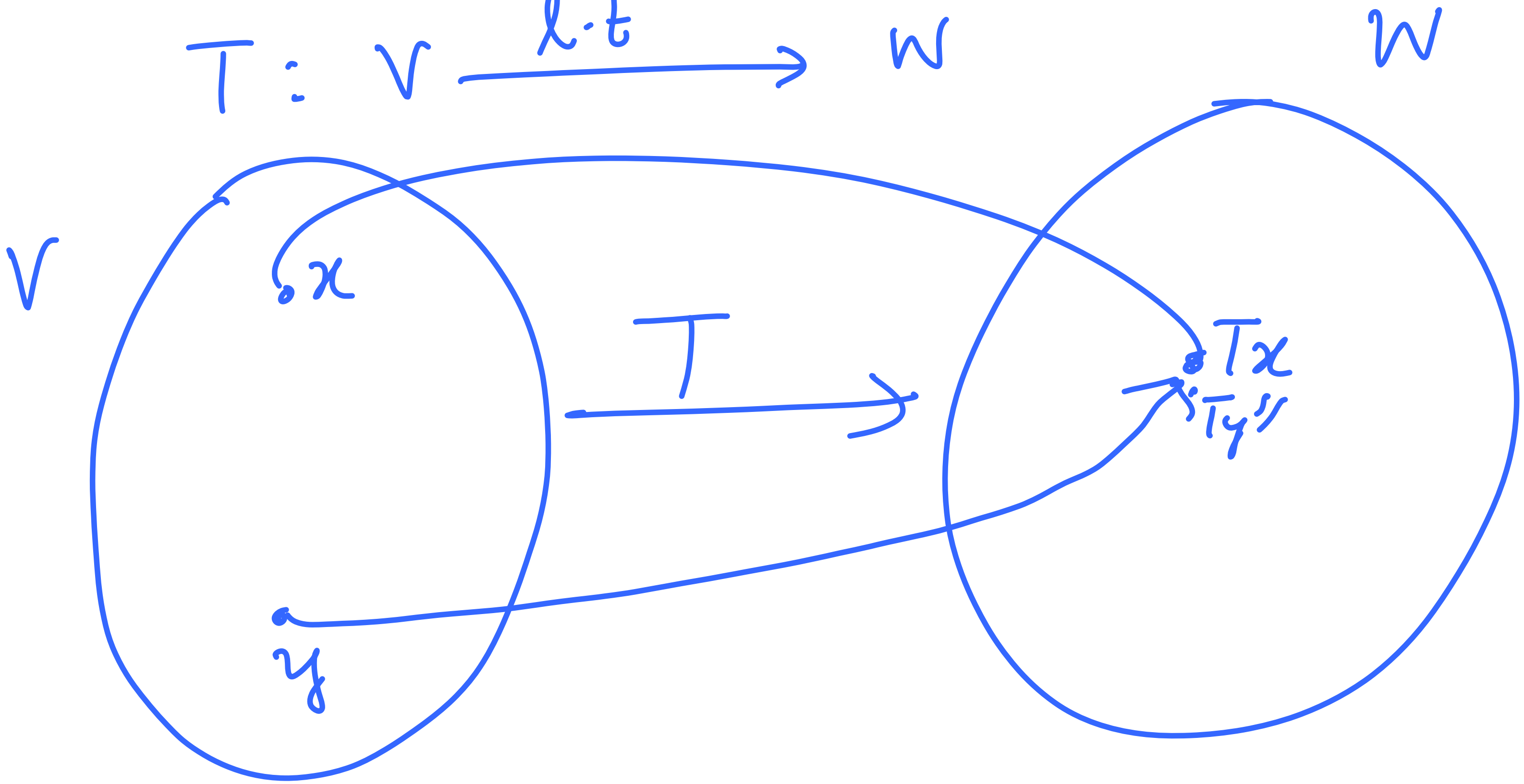
$$\rho_T + \nu_T = 3 + 2 = 5 = \dim \mathbb{F}^4$$
$$= \dim V$$

$$\boxed{\text{RANK} + \text{NULLITY} = \text{DIM } V}$$

V, W f.d.v.s over $\mathbb{F}$

$\dim V = n \quad \dim W = m$

$T : V \xrightarrow{l.t} W$



T can be thought of as a coding of V vectors as W vectors

Is this a good coding?

In order that different vectors in V have different coded versions we need,

$$Tx = Ty \implies x = y$$

This leads to the following:

<u>Def</u>: Let V and W be vector spaces over $\mathbb{F}$

$T: V \longrightarrow W$ a l.t

is said to be one-one if

$$Tx = Ty \implies x = y$$

(same as asking $x \neq y \Rightarrow Tx \neq Ty$)

Suppose T is one-one l.t.

$$x \in N_T \implies Tx = \theta_W$$

On the other hand

$T(\theta_V) = \theta_W$ since T is l.t.

We get therefore

$$Tx = T\theta_V$$

$\implies x = \theta_V$ since T is one-one

Hence $N_T = \{\theta_V\}$

$\nu_T = \dim N_T = 0$

Hence

T is one-one $\Longrightarrow \nu_T = 0$ (i.e. $N_T = \{\theta_V\}$)

If V and W are also f.d.v.s & T is one-one from V to W, by rank nullity theorem we have

rank T + nullity $T = \dim V$

$\Rightarrow$ rank $T = \dim V$ if T is one-one

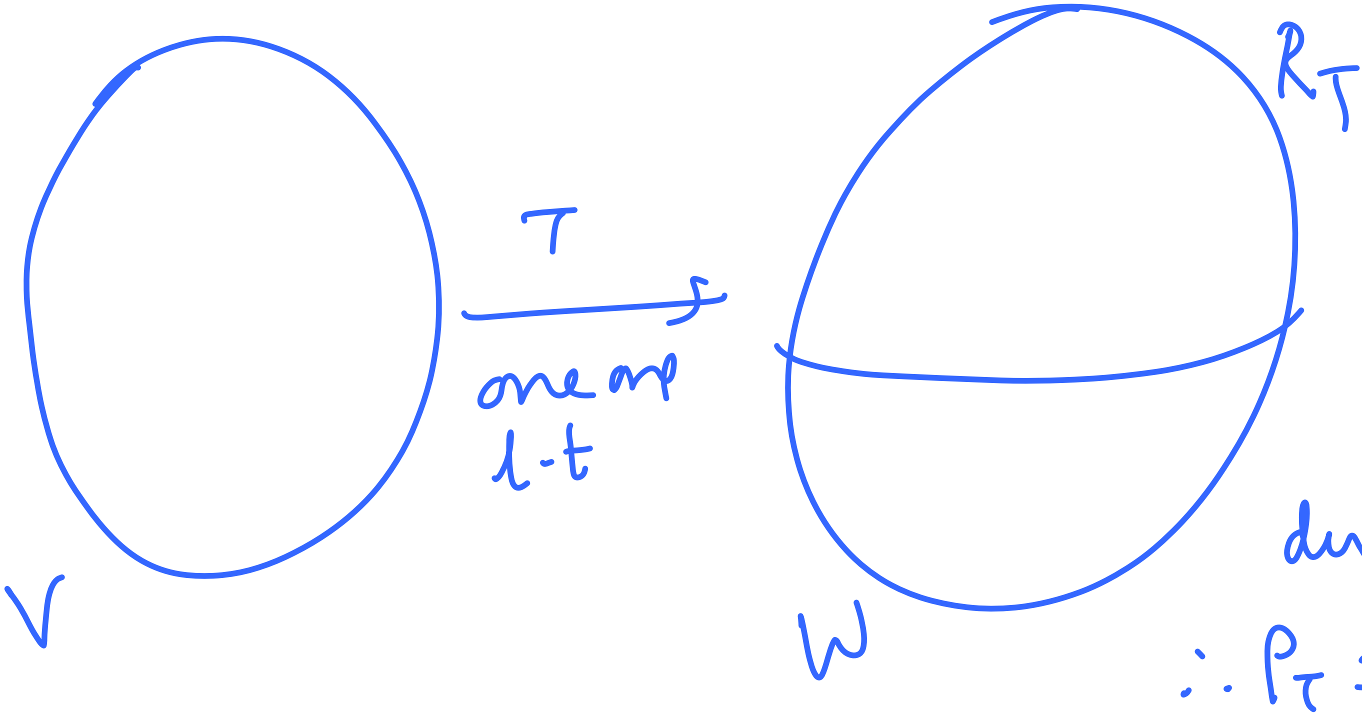
V, W fdvs

$T : V \longrightarrow W$ one one

$\implies$ i) $N_T = \{\theta_V\}$

ii) $\nu_T = 0$

iii) $\rho_T = \dim V$



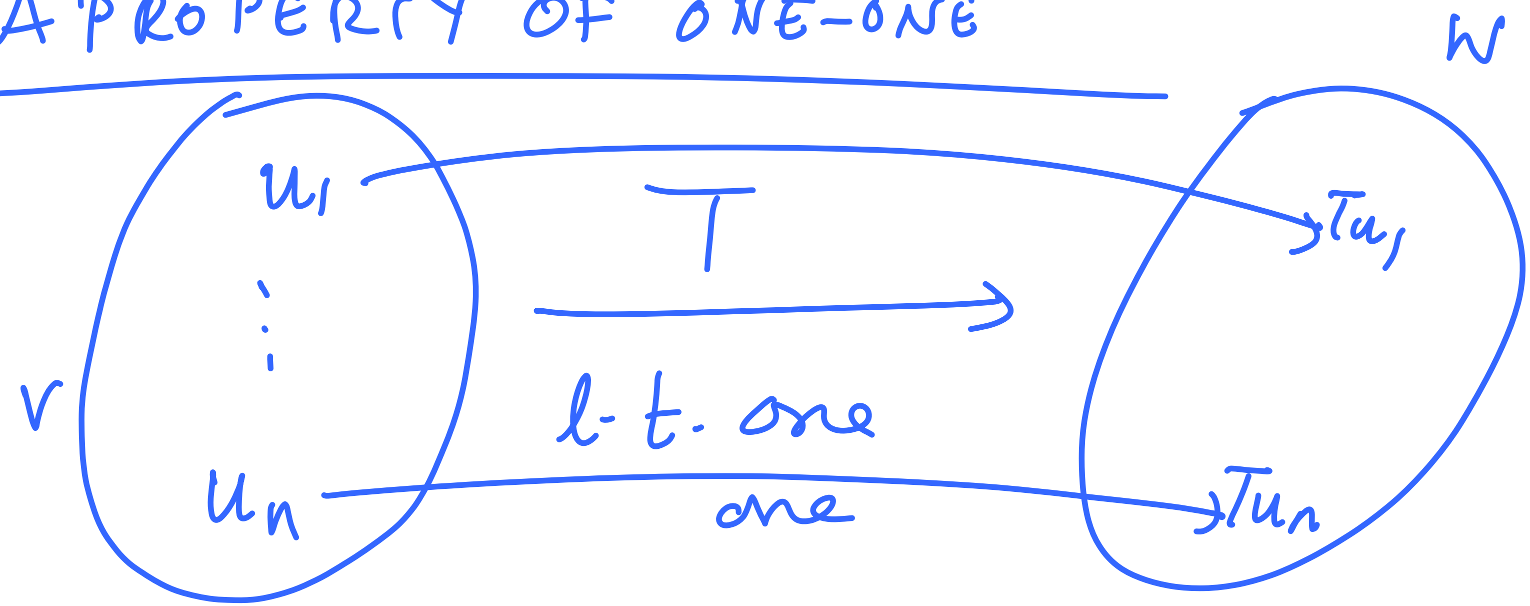
$\dim R_T \leq \dim W$

$\therefore \rho_T \leq \dim W$

Hence $\dim V \le \dim W$

Hence WE CANNOT HAVE A ONE-ONE LINEAR TRANSFORMATION FROM A VECTOR SPACE TO A LOWER DIMENSIONAL VECTOR SPACE

A PROPERTY OF ONE-ONE



$\dim V = n$ $\dim W = m$

$n \le m$

If $u_1, \ldots, u_r$ l.i. set in V

then can we conclude that

$Tu_1, Tu_2, \ldots, Tu_r$ is l.i in W