$$T : V \longrightarrow W \quad (\text{over } \mathbb{F})$$

$\quad$ fdvs $\qquad\qquad\qquad$ fdvs

R_T : Range of T

$\mathcal{N}_T$: Null space T

$$\mathcal{N}_T = \{ x \in V : Tx = \theta_W \}$$

$$R_T = \{ b \in W : \exists\, x \in V \ni Tx = b \}$$

$\dim R_T$: rank of T, ρ_T

$\dim N_T$: nullity of T, ν_T

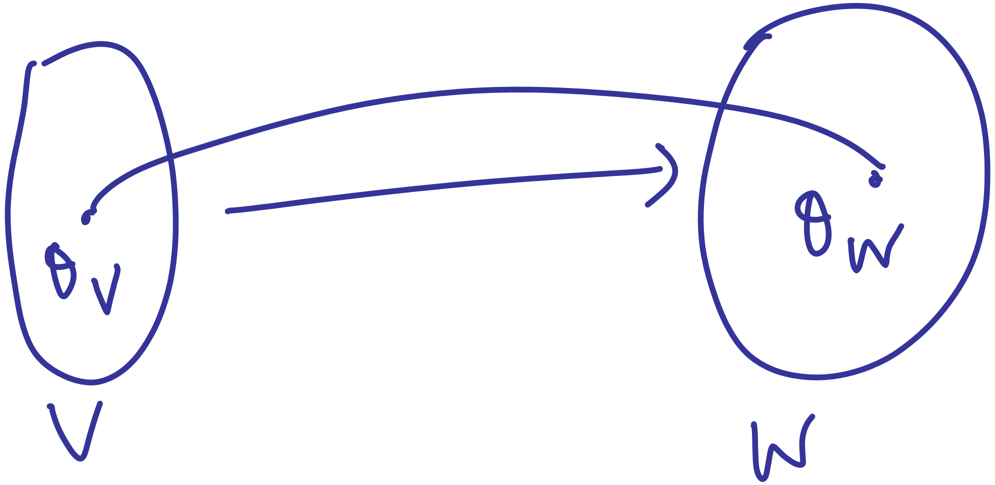
Rank- Nullity Theorem

$$\boxed{\rho_T + \nu_T = \dim V}$$

<u>One- One l.t</u>

V, W f.d.v.s over $\mathbb{F}$

T :



T l.t

We say T is one-one if

$$Tx = Ty \implies x = y$$

Suppose T is one one

Then $Tx = \theta_W$

$$\implies x = \theta_V$$

T is one one $\Rightarrow \mathcal{N}_T = \{\theta_V\}$

$\Rightarrow \dim \mathcal{N}_T = 0$

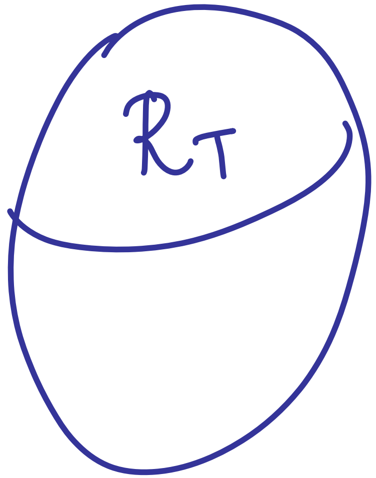
$\Rightarrow \nu_T = 0$

$\Rightarrow \nu_T + \rho_T = \dim V$ (Rank Nullity Thm)

$\Rightarrow \rho_T = \dim V$

$\left.\begin{array}{l}T \text{ is a l.t. one one} \\ \text{from } V \text{ to } W\end{array}\right\} \Rightarrow \rho_T = \dim V$

R_T is a subspace of W



W

$$\dim R_T \le \dim W$$

$$\Rightarrow \dim V \le \dim W$$

Thus a priori requirement to have a one-one l.t from V to W is that $\dim V \le \dim W$

Examples:

1) $V = \mathbb{R}^3$; $W = \mathbb{R}^3$

$T : V \longrightarrow W \ (= V)$

$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ as

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

Easy to check that T is linear tr.

We shall check T is one one

To check if $Tx = Ty \Rightarrow x = y$.

$$Tx = Ty \implies \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_1 \end{pmatrix}$$

$$\implies x_2 = y_2, \quad x_3 = y_3, \quad x_1 = y_1$$

$$\implies x = y$$

Hence T is one-one

<u>Ex 2</u>: $V = \mathbb{R}^3$ $\quad W = \mathbb{R}^4$

$$T : V \longrightarrow W$$

i.e. $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ as

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 + x_2 + x_3 \end{pmatrix}$$

Easy to see T is a l.t.

Is T one-one?

To check if $Tx = Ty \Rightarrow x = y$

$$Tx = Ty \Longrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_1+x_2+x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_1+y_2+y_3 \end{pmatrix}$$

$$\Longrightarrow x_1 = y_1,\ x_2 = y_2,\ x_3 = y_3$$

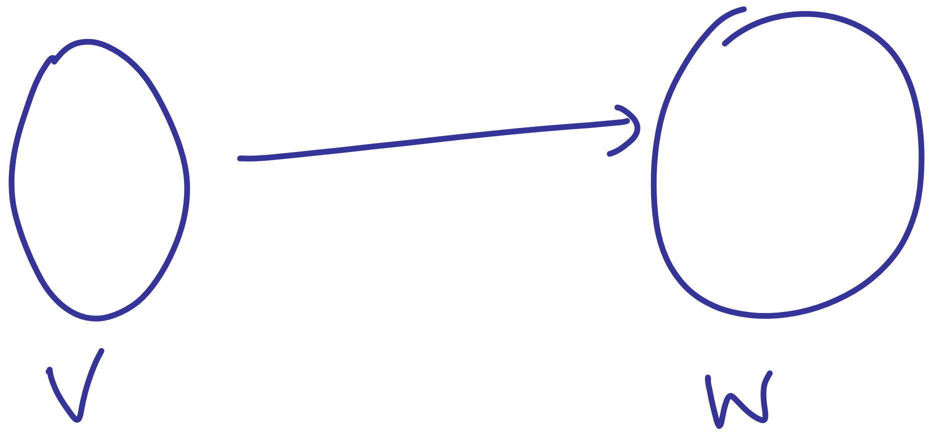
$$\Longrightarrow x = y$$

Hence T is one-one

Onto l t

V, W f d. v.s. over $\mathbb{F}$

$T:$



T l.t

We say T is onto if

$\forall w \in W \quad \exists v \in V \ni Tv = w$

i.e. if $R_T = W$

T is onto if $R_T = W$

T is onto $\Rightarrow R_T = W$

$\Rightarrow \dim R_T = \dim W$

$\Rightarrow \rho_T = \dim W$

$$\boxed{T \text{ is onto } \Rightarrow \rho_T = \dim W}$$

From Rank-Nullity Thm:

$$\rho_T + \nu_T = \dim V$$

$$\Rightarrow \boxed{\rho_T \leq \dim V}$$

T is onto $\Rightarrow \dim W \leq \dim V$

á priori requirement to have an Onto Transf is that

$$\dim W \leq \dim V$$

Example: $V = \mathbb{R}^3 = W$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

Clearly given any

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \in \mathbb{R}^3$$

if we define

$$v = \begin{pmatrix} w_3 \\ w_1 \\ w_2 \end{pmatrix}$$

then we get

$$T(v) = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$\forall\, w \in W \;\exists\, v \in V \;\ni\; Tv = w$

$\Longrightarrow R_T = W$

$\Longrightarrow$ T is onto

<u>Example</u>: $V = R^4 \quad W = R^3$

$T : V \longrightarrow W$ as

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Clearly it is onto since for any

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in W$$ we can take

$$y = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 0 \end{pmatrix} \in V$$ and

get $T(y) = x$

$\Rightarrow R_T = W \Rightarrow T$ is onto

Recall

A Priori Requirement to have
one-one l-t : $\dim V \le \dim W$
for onto to exist : $\dim W \le \dim V$

Hence a priori requirement to have
a l.t from V to W which is both
one-one and onto is that
$$\dim V = \dim W$$

Example: $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}$$

We found T is both one-one and onto

Definition:

Let V & W be f.d.v.s. over $\mathbb{F}$

A l.t. $T: V \longrightarrow W$ is called an ISOMORPHISM if

V onto W if T is both one-one and onto.

If such a T exists then we say V is <u>isomorphic</u> to W

Suppose V is isomorphic to W

This $\Rightarrow$ $\exists$ $T: V \longrightarrow W$ s.t.

T is one one & onto

$\Rightarrow$ $\dim V = \dim W$

Hence

V is isomorphic to W

$\Longrightarrow \dim V = \dim W$

Hence can not expect two f-d-v.s to be isomorphic if their dimensions are not equal

Question If $\dim V = \dim W$ then is V necessarily isomorphic to W?

A Simple Property:

$$\dim V = \dim W$$

$T: V \longrightarrow W$ l.t

$$
\begin{aligned}
T \text{ is onto} &\iff R_T = W \\
&\iff \rho_T = \dim W \\
&\iff \rho_T = \dim V \\
&\iff \nu_T = 0 \quad \text{(by Rank-Nullity Theorem)} \\
&\iff N_T = \{\theta_V\} \\
&\iff T \text{ is one-one}
\end{aligned}
$$

Hence to check if T is an isomorphism from V onto W it is enough to check if T is onto or if T is one-one

Recall the question

If $\dim V = \dim W$ is V necessarily isomorphic to W?

i.e if $\dim V = \dim W$ can we find a l.t. $T : V \longrightarrow W$ s.t

T is one-one and onto

Let us investigate this:

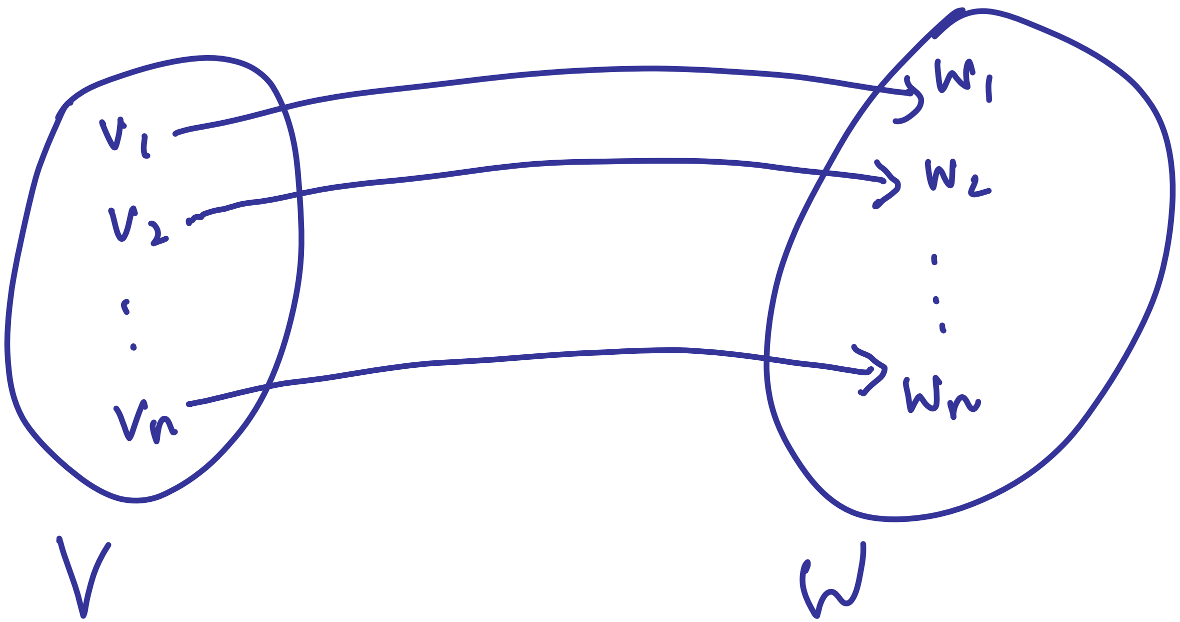
Let $\dim V = \dim W = n$

Let us choose any ordered bases

$\beta_V : v_1, v_2, \dots, v_n$ for V

Let us choose any ordered basis

$\beta_W : w_1, w_2, w_3, \dots, w_n$ for W



First we define T as

$$T(v_1) = w_1,\ T(v_2) = w_2,\ \dots,\ T(v_n) = w_n$$

Suppose $x \in V$

Then x can be written as a unique l.c.

$$x = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n$$

of the basis vectors in B_V

Then

$$Tx = T(x_1 v_1 + x_2 v_2 + \cdots + x_n v_n)$$

$$= T(x_1 v_1) + T(x_2 v_2) + \cdots + T(x_n v_n)$$

(since we want to be linear)

$$= x_1 T(v_1) + x_2 T(v_2) + \cdots + x_n T(v_n)$$

$$= x_1 w_1 + x_2 w_2 + \cdots + x_n w_n$$

Thus we have the def of T as

$$T(V_j) = W_j \,, \quad j = 1, 2, \cdots, n$$

For any $x = x_1 v_1 + x_2 v_2 + \cdots + x_n v_n \in V$

$$Tx \overset{def}{=} x_1 w_1 + x_2 w_2 + \cdots + x_n w_n$$

<u>Claim</u>: 1) T is a linear transf ✓

2) T is isomorphism

To show T is isomorphism we have to show T is one-one and onto & since dim V = dim W it is enough to check if T is onto

We have

$w_1, w_2, \ldots, w_n \in R_T$

Hence $\dim R_T \geq n$

$R_T \subseteq W =$ (has dim n)

$\dim R_T \leq n$

$\Rightarrow \quad \dim R_T = n = \dim W$

$\Rightarrow \quad R_T = W$

$\Rightarrow$ T is onto

Hence T is an isomorphism

Summary

$\dim V = \dim W$

$= n$

Any pair of bases

$\beta_V = v_1, \dots, v_n$ for V

$\beta_W = w_1, \dots, w_n$ for W

leads to an isomorphism of V onto W as defined above

Thus we have

Any two V.S. V, W over $\mathbb{F}$ having same dimension are isomorphic and every pair of basis B_V for V & B_W for V leads to an isomorphism of V onto W

$\mathbb{F}$. field

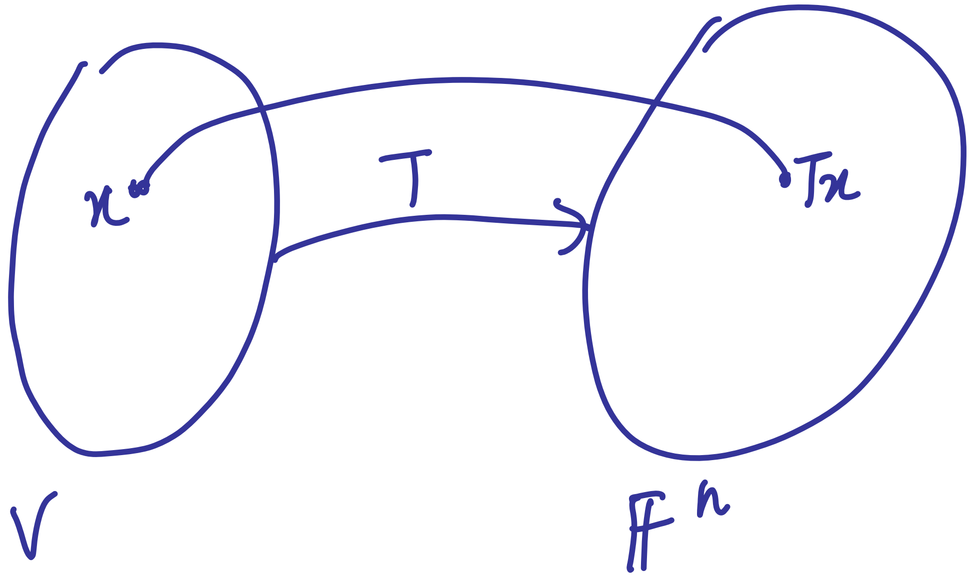
We know that $\mathbb{F}^n$ is an n dim. vector space over $\mathbb{F}$

Consider any n dim. vector space V over $\mathbb{F}$.

$$\dim V = n = \dim \mathbb{F}^n$$

$\Rightarrow$ V must be isomorphic to $\mathbb{F}^n$

$\Rightarrow$ $\exists \; T : V \longrightarrow \mathbb{F}^n$ which is linear, one-one & onto



T comes from $\mathcal{B}_V$ for V

$\mathcal{B}_{\mathbb{F}^n}$ for $\mathbb{F}^n$

$x \in V$ is coded as $Tx \in \mathbb{F}^n$

This code is one-one & onto

$\Rightarrow$ Instead of working in V
we can work in $\mathbb{F}^n$

Then finally we can translate
back to V through T^{-1} (which
exists since T is one-one onto)

Thus the only "meaningful" n dimensional
v.sp. over $\mathbb{F}$ is $\mathbb{F}^n$ and any other vector
space of dim n over $\mathbb{F}$ is essentially $\mathbb{F}^n$

Spoken in another language.