$\mathbb{F}$ field

Then any n dimensional vect. space over $\mathbb{F}$ is isomorphic to $\mathbb{F}^n$

In particular if we take $\mathbb{F} = \mathbb{R}$ then the basic vect sp of dimension k is $\mathbb{R}^k$.

Hence we look at $\mathbb{R}^k$ in more detail

Dot Product

$\mathbb{R}^3$ $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \ y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

we define <u>dot product</u>

$$x \cdot y \overset{\text{def}}{=\!=} x_1 y_1 + x_2 y_2 + x_3 y_3$$

<u>Generalize:</u>

$\mathbb{R}^k$ $\quad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{pmatrix}, \; y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{pmatrix},$

we define Inner Product of x with y as

$$(x, y) \overset{\text{def}}{=\!=} x_1 y_1 + x_2 y_2 + \cdots + x_k y_k$$

$$= y^T x$$

Simple Properties

1. $(x, x) = \sum_{j=1}^{k} x_j^2$

$\geq 0 \quad \forall x \in \mathbb{R}^k, \quad = 0 \text{ iff } x = \theta_k$

2. $(x, y) = (y, x) \quad \forall x, y \in \mathbb{R}^k$

3. $(x+y, z) = \sum_{j=1}^{k} (x_j + y_j) z_j$

$= \sum_{j=1}^{k} x_j z_j + \sum_{j=1}^{k} y_j z_j$

$= (x, z) + (y, z) \quad \forall x, y, z \in \mathbb{R}^k$

(Inner Product is Right Distributive)

4. $(\alpha x, y) = \sum_{j=1}^{k} (\alpha x_j) y_j$

$$= \alpha \sum_{j=1}^{k} x_j y_j$$

$$= \alpha \, (x, y) \qquad \forall \, \alpha \in \mathbb{F}, \; x, y \in \mathbb{R}^k$$

Remark:

Using (2) in (3) and (4) we get

$$(x, y+z) = (x, y) + (x, z) \quad \forall \, x, y, z \in \mathbb{R}^k$$

(left destributive)

$$(x, \alpha y) = \alpha (x, y) \quad \forall \, \alpha \in \mathbb{F}, \; x \in \mathbb{R}^k, \; y \in \mathbb{R}^k$$

Effect Of Inner Product

$\mathbb{R}^3$

$$(x, y) = \sum_{j=1}^{3} x_j y_j$$

$$(x, x) = \sum_{j=1}^{3} x_j^2$$

$$\Rightarrow \quad \underset{\text{length of } x}{\|x\|} = \sqrt{(x, x)}$$

Generalize:

In $\mathbb{R}^k$ for any $x \in \mathbb{R}^k$ we define the length of x — denote by $\|x\|$ (call it as NORM of x) as

$$\|x\| \stackrel{\text{def}}{=\!=} \sqrt{(x, x)}$$

Properties

i) $\|x\| \geq 0 \ \forall x \in \mathbb{R}^k$ and $= 0$ iff $x = \theta_k$

ii) $\|\alpha x\| = |\alpha| \, \|x\| \ \forall \alpha \in \mathbb{F}, \ \forall x \in \mathbb{R}^k$

iii) $\|x+y\| \leq \|x\| + \|y\| \ \forall x, y \in \mathbb{R}^k$

(Triangle Inequality)

— (Follows Cauchy-Schwarz Inequality)

Orthogonality

In $\mathbb{R}^3$ $\quad x, y \in \mathbb{R}^3$ we say $x \perp y$

if $x \cdot y = 0$

Generalize:

If $x, y \in \mathbb{R}^k$ we say x is ORTHOGONAL to y iff $(x, y) = 0$

$\left((x,y) = 0 \iff (y,x) = 0 \right.$ Hence x orth. to y $\left. \iff y \text{ orth to } x \right)$

For any $x \in \mathbb{R}^k$

Note: $(x, \theta_k) = \sum_{j=1}^{k} x_j (\theta_k)_j = 0$

$\Rightarrow \theta_k$ is orthogonal to all the vectors in $\mathbb{R}^k$

In fact, the zero vector θ_k is the ONLY vector orthogonal to all the vectors in $\mathbb{R}^k$

Why? Suppose $x \in \mathbb{R}^k$ and x orth. to all vect

This $\Rightarrow$ x orth to x

$\Rightarrow (x,x)=0$

$\Rightarrow x = \theta_R$

Examples

1. $\mathbb{R}^4$

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$(x,y) = 1-1+1-1 = 0$

$\Rightarrow$ x is orth to y in $\mathbb{R}^4$

2. $\mathbb{R}^3$ $\quad e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Clearly e_1 orth e_2 $\quad$ e_3 orth e_1

e_1 orth e_3

3 $\mathbb{R}^4$ $\quad x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Let us find all vectors in $\mathbb{R}^4$ which are orthogonal to x

$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ orth to x

$\iff (x, u) = 0$

$\iff u_1 + u_2 + u_3 + u_4 = 0$

$\iff u$ must be of the form

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ -\alpha-\beta-\gamma \end{pmatrix}, \ \alpha, \beta, \gamma \in \mathbb{R}$$

Hence

$$\left\{ u = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ -\alpha-\beta-\gamma \end{pmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\}$$

is the set of all the vectors orthogonal to x

Generalize this example.

Let S be any nonempty subset of $\mathbb{R}^k$

We denote by $S^\perp$ the set of all vectors in $\mathbb{R}^k$ which are orthogonal to all the vectors in S.

$$S^\perp = \{u \in \mathbb{R}^k : (s,u) = 0 \ \forall s \in S\}$$

i) Clearly $\theta_k \in S^\perp$ since θ_k is orthogonal to all vectors in $\mathbb{R}^k$ and hence in particular to all vectors in S

Now since $S^\perp$ is a nonempty subset of $\mathbb{R}^k$ we check if it is a subspace of $\mathbb{R}^k$. For this we have to check if

i) $x, y \in S^\perp \implies x + y \in S^\perp$ and

ii) $x \in S^\perp, \alpha \in \mathbb{R} \implies \alpha x \in S^\perp$

(i) $x, y \in S^\perp \implies (s,x) = 0 \ \forall s \in S, (s,y) = 0 \ \forall s \in S$

$\Rightarrow (s,x) + (s,y) = 0 \quad \forall s \in S$

$\Rightarrow \left(s, x+y\right) = 0 \quad \forall s \in S$

$\Rightarrow$ $x+y$ orth to all vect. in S

$\Rightarrow$ $x+y \in S^{\perp}$

Hence $S^{\perp}$ is closed under addition

ii) $x \in S^{\perp}, \alpha \in \mathbb{R} \Rightarrow (s,x) = 0 \quad \forall s \in S; \alpha \in \mathbb{R}$

$\Rightarrow \alpha(s,x) = 0 \quad \forall s \in S$

$\Rightarrow (s, \alpha x) = 0 \quad \forall s \in S$

$\Rightarrow$ αx orth to all vect in S

$\Rightarrow$ $\alpha x \in S^{\perp}$

$\Rightarrow$ $S^{\perp}$ is closed under scalar multiplication

Hence $S^\perp$ is a nonempty subset of $\mathbb{R}^k$ which is closed under addition and scalar multiplication

$\Rightarrow$ $S^\perp$ is a subspace of $\mathbb{R}^k$

Note: We have not assume S to be a subspace — but still $S^\perp$ is a subspace

The $S^\perp$ for any nonempty subset S in $\mathbb{R}^k$ is always a subspace of $\mathbb{R}^k$ irrespective of whether S is a subspace or not

In particular if W is a subspace

of $\mathbb{R}^k$ then $W^\perp$ will also be a subspace of $\mathbb{R}^k$

<u>Example:</u> $\mathbb{R}^3$

$$S = \{u_1, u_2\} \text{ where } u_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \; u_2 = \begin{pmatrix}1\\2\\3\end{pmatrix}$$

Note S is not a subspace

Let us find $S^\perp$

$x \in S^\perp \iff x$ is orth to u_1 <u>and</u> u_2

$\iff (u_1, x) = 0$ <u>and</u> $(u_2, x) = 0$

$\iff x_1 + x_2 + x_3 = 0$ -- (1) and

$x_1 + 2x_2 + 3x_3$ -- (2)

$\Longleftrightarrow$ $\begin{cases} (2)-(1) \text{ gives } & x_2 + 2x_3 = 0 \\ & \boxed{x_2 = -2x_3} \\ \text{use in } (1) & \\ x_1 - 2x_3 + x_3 = 0 & \boxed{x_1 = x_3} \end{cases}$

$\Longleftrightarrow$ x is of the form

$$x = \begin{pmatrix} \alpha \\ -2\alpha \\ \alpha \end{pmatrix} ; \alpha \in \mathbb{R}$$

$$\Rightarrow \quad S^{\perp} = \left\{ x = \begin{pmatrix} \alpha \\ -2\alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

Example: W subspace of $\mathbb{R}^4$

where $W = \left\{ \begin{pmatrix} \alpha \\ \beta \\ \alpha+\beta \\ \alpha-\beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$

What is $W^\perp$?

$x \in W^\perp$. Now a basis for W is

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

Now any vector in W is of the form $\alpha u_1 + \beta u_2$

Suppose x orth to u_1 and u_2

then $(x, u_1) = 0$, $(x, u_2) = 0$

$\Rightarrow (x, \alpha u_1) = 0$, $(x, \beta u_2) = 0$

$\Rightarrow (x, \alpha u_1) + (x, \beta u_2) = 0$

$\Rightarrow (x, \alpha u_1 + \beta u_2) = 0$

$\Rightarrow$ x orth to all vect in W

$x \in W^\perp \iff x$ Orth to u_1 & u_2

$$\iff \begin{aligned} x_1 + x_3 + x_4 &= 0 \\ x_2 + x_3 - x_4 &= 0 \end{aligned}$$

$$\iff \begin{aligned} x_1 &= -x_3 - x_4 \\ x_2 &= -x_3 + x_4 \end{aligned}$$

$$W^\perp = \left\{ u = \begin{pmatrix} -\alpha - \beta \\ -\alpha + \beta \\ \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

Basis for $W^{\perp}$

$$v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} ; \quad v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\dim W^{\perp} = 2$$

Orthonormal Sets

$\mathbb{R}^3$: $\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad \vec{j} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\quad \vec{k} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

any two distinct vect in this collection are orthogonal & every vect has length 1

In $\mathbb{R}^k$ a set

$$S = u_1, u_2, \dots, u_k$$

of vectors is said to be ORTHONORMAL set if

$$(u_i, u_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

(1st condition says
any two distinct vectors in S are orthogonal,
2nd condn says
any vector in S has length 1