Orthonormality in $\mathbb{R}^k$

Orthonormal sets

Orthormal basis

$\mathcal{B} : \varphi_1, \varphi_2, \dots, \varphi_k$ orthonormal basis for $\mathbb{R}^k$

Any $x \in \mathbb{R}^k$ can be expanded (Fourier expansion) in terms of $\mathcal{B}$ as

$$x = \sum_{j=1}^{k} \underline{(x, \varphi_j)}\, \varphi_j$$

(Recall $(x, \varphi_j) = \varphi_j^T x$)

Orthogonal Complement

W subspace of $\mathbb{R}^k$

$W^\perp$ is the collection of all those vectors in $\mathbb{R}^k$ which are orthogonal to all the vectors in W

$$W^\perp = \{ x \in \mathbb{R}^k : (x, w) = 0 \ \forall w \in W \}$$

We saw that $W^\perp$ is a subspace of $\mathbb{R}^k$

This subspace $W^\perp$ is called the ORTHOGONAL COMPLEMENT of W

Simple Observation

W : subspace of $\mathbb{R}^k$ $\quad \dim W = d$

$W^\perp$: orthogonal complement of W

Suppose B_W : $\underline{u_1, u_2, \cdot, u_d}$

is a basis for W

$x \in W^\perp \implies$ x is orthogonal to all the vectors in W

$\implies$ In particular x is orthogonal to all the vectors in the basis B_W

$\implies$ $(x, u_j) = 0$ for $j = 1, 2, \cdot, d$

Conclusion 1 $\quad x \in W^{\perp} \Rightarrow$ x is orthogonal to all the vectors in a basis for W

Conversely

Suppose $x \in \mathbb{R}^k$ is s.t.

x is orth. to all vectors in the basis $\mathcal{B}_W$

$$\Longrightarrow \underline{(x, u_j) = 0 \text{ for } j = 1, 2, \ldots, d}$$

Now take any $w \in W$

w can be expressed as a l.c. of the vectors in the basis $\mathcal{B}_W$ for W

We have

$$w = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_d u_d$$

$$\Rightarrow (x, w) = (x, \alpha_i u_1 + \alpha_2 u_2 + \cdots + \alpha_k u_k)$$

$$= \alpha_1 (x, u_1) + \alpha_2 (x, u_2) + \cdots + \alpha_d (x, u_d)$$

$$= 0$$

$\Rightarrow$ x is orthogonal to w for every $w \in W$

Thus $x \in W^\perp$

Conclusion 2

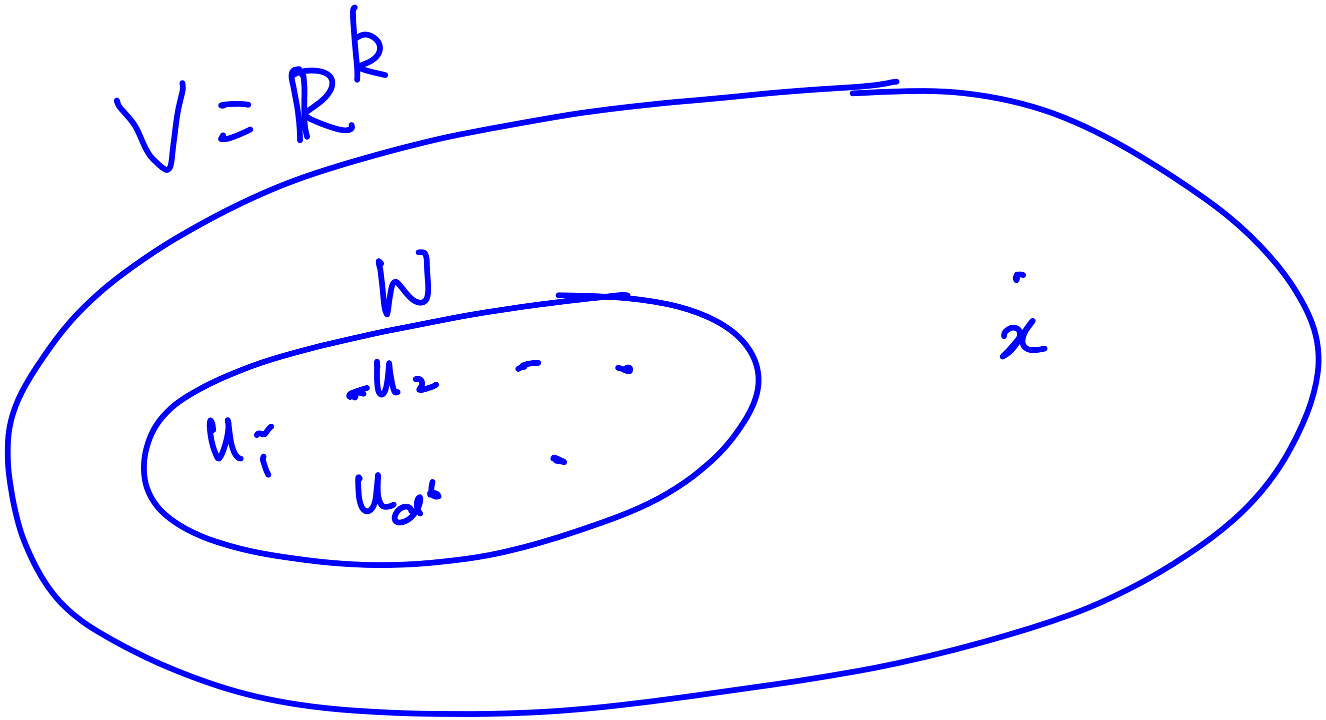
$(x, u_j) = 0$, $j = 1, 2, \ldots, d$

$\Rightarrow x \in W^\perp$

Combining Conclusion 1 and 2 we get

$x \in W^{\perp} \iff x$ is orthogonal to all the vectors in a basis for W

? $x \in W^{\perp}$



Example

(1) $\mathbb{R}^3$

$$W = \left\{ x = \begin{pmatrix} a \\ a \\ a \end{pmatrix} : a \in \mathbb{R} \right\}$$

It is easy to check that W is a subspace of $\mathbb{R}^k$.

Let us find $W^\perp$

Clearly $u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ forms a basis for W

$$\boxed{\mathcal{B}_W : \quad u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

$\dim W = 1$

$$x \in W^\perp \iff (x, u) = 0$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \iff x_1 + x_2 + x_3 = 0$$

$\iff x$ is of the form $\begin{pmatrix} \alpha \\ \beta \\ -\alpha-\beta \end{pmatrix}$ for $\alpha, \beta \in \mathbb{R}$

$$W^\perp = \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ -\alpha-\beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

Note that $\boxed{\mathcal{B}_{W^\perp} : v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$

is a basis for $\mathcal{B}_{W^\perp}$ $\qquad \dim W^\perp = 2$

$$\boxed{\dim W + \dim W^{\perp} = 1+2 = 3 = \dim \mathbb{R}^3}$$

Note also that $\boxed{\mathcal{B}_W \cup \mathcal{B}_{W^{\perp}} = \mathcal{B}}$

$$u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

This is l.i. & has 3 vectors & $\therefore$ a basis for $\mathbb{R}^3$

The union of a basis for W and a basis for $W^{\perp}$ is a basis for $\mathbb{R}^3$

Example 2 $\mathbb{R}^4$

$$W = \left\{ x = \begin{pmatrix} a \\ b \\ a+b \\ a-b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

It is easy to check that W is a subspace of $\mathbb{R}^4$.

Let us find $W^{\perp}$

It is easy to see that

$$\mathcal{B}_W : u_1 = \underline{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}}, \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

is a basis for W

$$\underline{\dim W = 2}$$

$$x \in W^{\perp} \iff \underline{(x, u_1) = 0} \text{ and } \underline{(x, u_2) = 0}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \iff \begin{matrix} x_1 + x_3 + x_4 = 0 \text{ and} \\ x_2 + x_3 - x_4 = 0 \end{matrix}$$

$\iff$ x is of the form

$$\begin{pmatrix} -\alpha - \beta \\ -\alpha + \beta \\ \alpha \\ \beta \end{pmatrix}, \alpha, \beta \in \mathbb{R}$$

$$W^{\perp} = \left\{ x = \begin{pmatrix} -\alpha - \beta \\ -\alpha + \beta \\ \alpha \\ \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

$B_{W^\perp}$: $v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

is a basis for $W^\perp$

$$\dim W^\perp = 2$$

$$\dim W + \dim W^\perp = 2+2 = 4 = \dim \mathbb{R}^4$$

<u>Check</u> $B_W \cup B_{W^\perp}$

$= u_1, u_2, v_1, v_2$

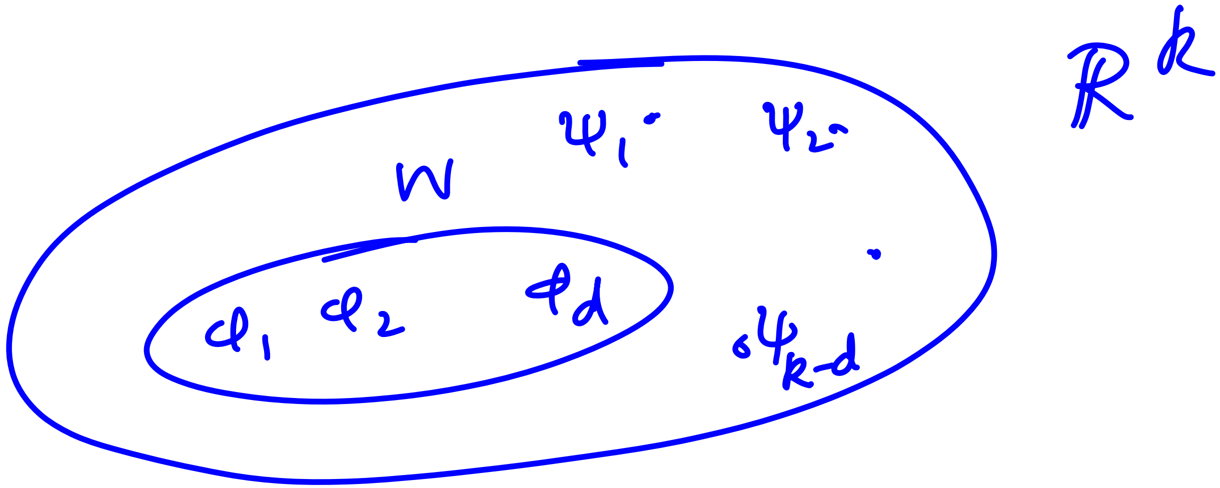
These are l.i & $\therefore$ a basis for $\mathbb{R}^4$

$\mathbb{R}^k$ $\quad W$ a subspace of $\mathbb{R}^k$

$\dim W = d$

Let $\mathcal{B}_W : \varphi_1, \varphi_2, \ldots, \varphi_d$

be an orthonormal basis for W



We can extend $\mathcal{B}_W$ to an o.n.b. for $\mathbb{R}^k$

Let the extended basis be,

$$\mathcal{B} : \varphi_1, \varphi_2, \ldots, \varphi_d, \psi_1, \psi_2, \ldots, \psi_{k-d}$$

This being an o.n. basis for $\mathbb{R}^k$, any $x \in \mathbb{R}^k$ can be expanded as

$$x = (x,\varphi_1)\varphi_1 + \cdots + (x,\varphi_d)\varphi_d + (x,\psi_1)\psi_1 + \cdots + (x,\psi_{k-d})\psi_{k-d}$$

$$\boxed{x = \sum_{j=1}^{d} (x,\varphi_j)\varphi_j + \sum_{j=1}^{k-d} (x,\psi_j)\psi_j \quad \text{--} \quad (\mathrm{I})}$$

Some observations:

i) Note since $\mathcal{B}$ is an orthonormal basis for $\mathbb{R}^k$ every vector in it is orthogonal to every other vector in it

In particular

$$(\psi_1, \varphi_j) = 0 \quad \text{for } j = 1, 2, \ldots, d$$

$\Rightarrow$ ψ_1 is orthogonal to all the vectors in the Basis B_W for W

$\Rightarrow$ $\psi_1 \in W^\perp$

Similarly $\psi_2, \psi_3, \ldots, \psi_{k-d} \in W^\perp$

$\therefore$ We have

i) $\underline{\psi_1, \psi_2, \ldots, \psi_{k-d} \in W^\perp}$

ii) $\underline{\psi_1, \psi_2, \ldots, \psi_{k-d} \text{ is orthonormal}}$

Is this a basis for $W^\perp$

For this we need to check if this set spans $W^\perp$.

$$x \in W^\perp \Rightarrow (x, \varphi_j) = 0, \quad j = 1, 2, \dots, d$$

$\Rightarrow$ The expansion for x in terms of $\mathcal{B}$ by (I) becomes

$$x = \sum_{j=1}^{k-d} (x, \psi_j) \psi_j$$

$\Rightarrow$ x is a l.c. of $\psi_1, \dots, \psi_{k-d}$

Hence $\underline{\psi_1, \psi_2, \dots, \psi_{k-d} \text{ span } W^\perp}$

$\Rightarrow$ $\psi_1, \psi_2, \ldots, \psi_{k-d}$ an orthonormal basis for $W^\perp$

$$\dim W^\perp = k-d = \dim \mathbb{R}^k - \dim W$$

$\Rightarrow$

$$\boxed{\dim W + \dim W^\perp = \dim \mathbb{R}^k}$$

Let us look at (I)

$x \in \mathbb{R}^k$, $\quad x = \sum_{j=1}^{d} (x, \varphi_j)\varphi_j + \sum_{j=1}^{k-d} (x, \psi_j)\psi_j$

$$\boxed{x = x_W + x_{W^\perp}}$$

where

$$x_W = \sum_{j=1}^{d} (x, \varphi_j)\, \varphi_j \in W$$

$$x_{W^\perp} = \sum_{j=1}^{k-d} (x, \psi_j)\, \psi_j \in W^\perp$$

Thus

Every $x \in \mathbb{R}^k$ can be written as the sum of a vector $x_W \in W$ and a vector $x_{W^\perp} \in W^\perp$

Is this decomposition unique?

Suppose $x = x_W + x_{W^\perp}$

& $x = x'_W + x'_{W^\perp}$

$x_W, x'_W \in W$

$x_{W^\perp}, x'_{W^\perp} \in W^\perp$

$$\theta_k = (x_W - x'_W) + (x_{W^\perp} - x'_{W^\perp})$$

$$\Rightarrow \underbrace{(x'_W - x_W)}_{\in W} = \underbrace{(x_{W^\perp} - x'_{W^\perp})}_{\in W^\perp} = Z, \text{ say}$$

$\Rightarrow$ $Z \in W \cap W^\perp$

$\Rightarrow (Z, Z) = 0 \Rightarrow Z = \theta_k$

$\Rightarrow x_W = x'_W$ and $x_{W^\perp} = x'_{W^\perp}$

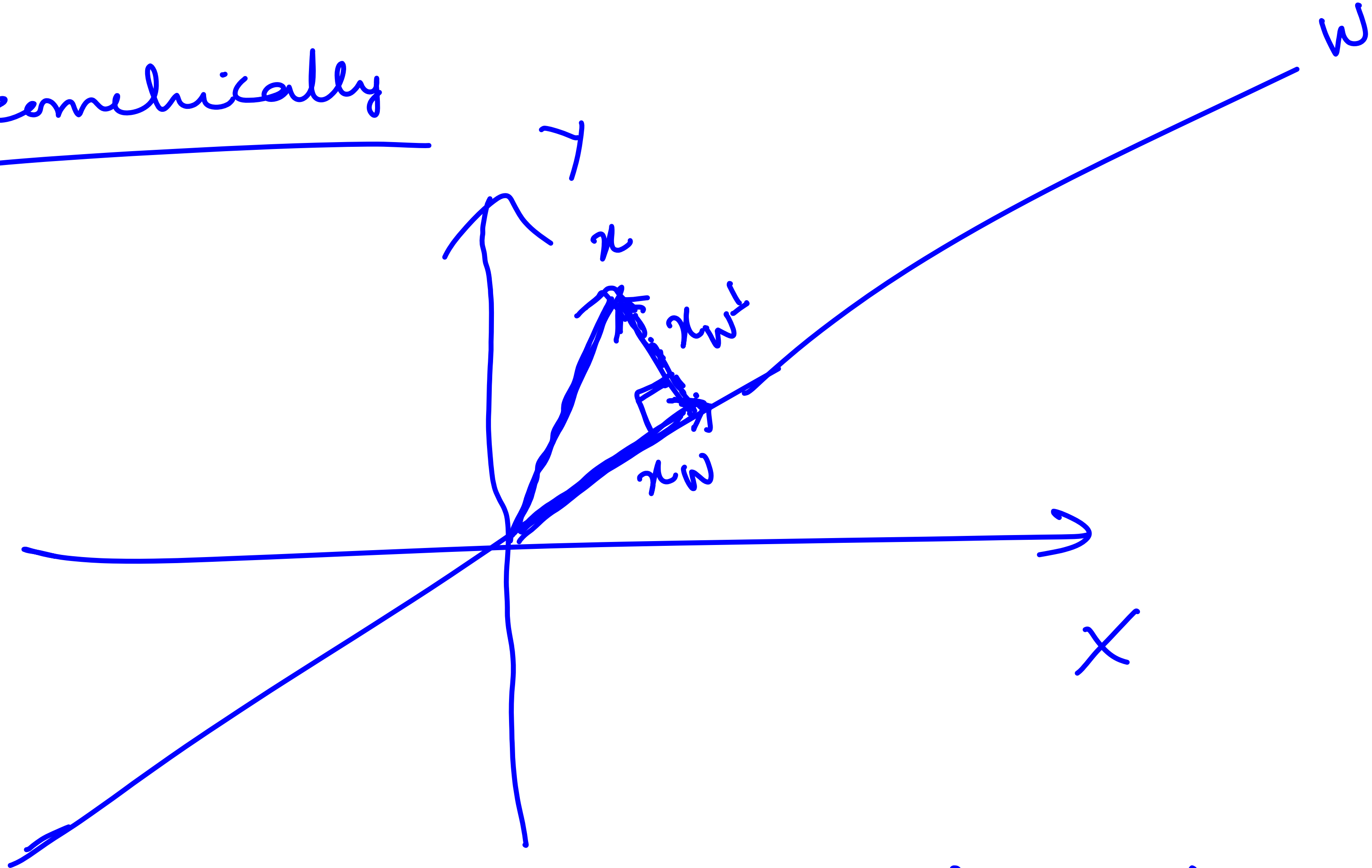
Theorem: Let W be any subspace of $\mathbb{R}^k$. Then every vector $x \in \mathbb{R}^k$ can be decomposed uniquely as the sum

$$x = x_W + x_{W^\perp}$$

where $x_W \in W$ and $x_{W^\perp} \in W^\perp$

The x_W is called the ORTHOGONAL PROJECTION of x onto W and $x_{W^\perp}$ is called the ORTHOGONAL PROJECTION of x onto $W^\perp$.

Geometrically



Pythagoras Thm $\|x\|^2 = \|x_W\|^2 + \|x_{W^\perp}\|^2$

In fact this theorem also gets generalized

Pythagoras Theorem

W subspace of $\mathbb{R}^k$

$W^\perp$ its orth. complement

$x \in \mathbb{R}^k$

$x = x_W + x_{W^\perp}$ as its decomp, $x_W \in W$

x_W: orth proj of x onto W $\quad x_{W^\perp} \in W^\perp$

$x_{W^\perp}$: " " " " " $W^\perp$

$$\boxed{\text{Then } \|x\|^2 = \|x_W\|^2 + \|x_{W^\perp}\|^2}$$

Proof:

$$\|x\|^2 = (x, x) = (x_W + x_{W^\perp}, x_W + x_{W^\perp})$$

$$= (x_W, x_W) + (x_W, x_{W^\perp}) + (x_{W^\perp}, x_W)$$
$$+ (x_{W^\perp}, x_{W^\perp})$$
$$= \|x_W\|^2 + 0 + 0 + \|x_{W^\perp}\|^2$$

$$\Rightarrow \boxed{\|x\|^2 = \|x_W\|^2 + \|x_{W^\perp}\|^2}$$