$A \in \mathbb{C}^{n \times n}$

$c_A(\lambda)$ monic poly of deg n with coeffts in $\mathbb{C}$

$= \det(\lambda I - A)$

The eigenvalues of A are precisely the roots of this polynomial $c_A(\lambda)$

(If $A \in \mathbb{R}^{n \times n}$ then the complex roots of $c_A(\lambda)$ must occur in conjugate pairs)

By Fundamental Theorem of Algebra gives us that $C_A(\lambda)$ will have n roots in $\mathbb{C}$ (some of them may be repeated)

Suppose $\lambda_1, \lambda_2, \dots, \lambda_k$ are the DISTINCT roots of $C_A(\lambda)$ with

λ_1 repeating a_1 times

λ_2 " a_2 "

- -

λ_k " a_k "

This means

$$\boxed{c_A(\lambda) = (\lambda-\lambda_1)^{a_1}(\lambda-\lambda_2)^{a_2}\cdots(\lambda-\lambda_k)^{a_k}}$$

where $a_1 + a_2 + \cdots + a_k = n$

and $a_1 \geq 1,\ a_2 \geq 1,\ \ldots,\ a_k \geq 1$

$\lambda_1, \ldots, \lambda_k$ the distinct eigenvalues of A

$a_1, \ldots, a_k$ are called the ALGEBRAIC MULTIPLICITIES of $\lambda_1, \ldots, \lambda_k$ respectively.

Example

$$A = \begin{pmatrix} 1 & -3 & 2 \\ -2 & 0 & 2 \\ 1 & -1 & 2 \end{pmatrix}$$

$$C_A(\lambda) = (\lambda-4)(\lambda-2)(\lambda+2)$$

$\lambda_1 = 4$; $a_1 = 1$

$\lambda_2 = 2$; $a_2 = 1$

$\lambda_3 = -2$; $a_3 = 1$

DISTINCT EIGENVALUES OF A are $4, 2, -2$

Each having Algebraic Multiplicity (am) one

Example

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

$$C_A(x) = (\lambda - 4)^2 (\lambda - 2)$$

$$\left.\begin{array}{ll} \lambda_1 = 4 \ ; & a_1 = 2 \\ \lambda_2 = 2 \ ; & a_2 = 1 \end{array}\right\}$$

Example: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$C_A(\lambda) = \det \lambda I - A$$

$$= \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = \lambda^2$$

$\lambda_1 = 0$, $a_1 = 2$

<u>Example</u> $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$C_A(\lambda) = \lambda^2 + 1$$
$$= (\lambda + i)(\lambda - i)$$

$\lambda_1 = i$, $a_1 = 1$

$\lambda_2 = -i$; $a_2 = 1$

Note: A is real
eigenvalues complex
occur in conjugate pairs

Eigenvectors

$A \in \mathbb{C}^{n \times n}$. Let the charact poly be

$$C_A(\lambda) = (\lambda - \lambda_1)^{a_1} \cdots (\lambda - \lambda_k)^{a_k}$$

where $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues with a.m. respectively as $a_1, a_2, \dots, a_k$

Let λ_j be one of the eigenvalues

This means $\exists\, u \neq \theta_n$ s.t

$$\underline{A u = \lambda_j u}$$

This means the Null Space of $(A - \lambda_j I)$

contains a nonzero vector u

Let

$$W_j = \text{Null Space of } (A - \lambda_j I)$$

$$= \left\{ x \in \mathbb{C}^n : (Ax - \lambda_j x) = \theta_n \right\}$$

$$= \left\{ x \in \mathbb{C}^n : Ax = \lambda_j x \right\}$$

W_j is nontrivial ($\because u \neq \theta_u$ $u \in W_j$)

$\dim W_j \geq 1$

W_j is called the <u>eigenspace</u> corresponding to the eigenvalue λ_j.

(Every nonzero vector in W_j is an eigenvector corresp to λ_j)

$\dim W_j$ is called the GEOMETRIC (gm) Multiplicity of λ_j and is denoted by g_j

$g_j = \dim W_j$

$\underline{\underline{g_j \geq 1}}$ for every eigenvalue λ_j

Eigenvalue	algebraic mult	geometric mult
λ_j	a_j	g_j
	$a_j \geq 1$	$\underline{g_j \geq 1}$
	$a_1 + a_2 + \cdots + a_k = n$	

Remark:

It can be shown that

For every eigenvalue λ_j of A

$$1 \leq g_j \leq a_j \quad , \quad \text{for } j = 1, 2, \cdots, k$$

Examples

$$A = \begin{pmatrix} 1 & -3 & 3 \\ -2 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix}$$

$$C_A(\lambda) = (\lambda-4)(\lambda-2)(\lambda+2)$$

$$\left.\begin{array}{ll} \lambda_1 = 4 & a_1 = 1 \\ \lambda_2 = 2 & a_2 = 1 \\ \lambda_3 = -2 & a_3 = 1 \end{array}\right\}$$

<u>Eigenspaces:</u>

W_1 = Null Space $(A - \lambda_1 I)$

= Null Space $(A - 4I)$

$$A - 4I = \begin{pmatrix} -3 & -3 & 3 \\ -2 & -4 & 2 \\ 1 & -1 & -1 \end{pmatrix}$$

$$(A - 4I)x = \theta_3$$

We get

$$W_1 = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is a basis for W_1

$$\dim W_1 = 1 \implies \underline{g_1 = 1}$$

<u>Similarly</u>

$$W_2 = \text{Null Space}(A - \lambda_2 I)$$

$$= \text{Null Space}(A - 2I)$$

$$(A-2I) = \begin{pmatrix} -1 & -3 & 3 \\ -2 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$(A-2I)x = \theta_3$$

$$W_2 = \left\{ \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : \beta \in \mathbb{R} \right\}$$

$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is a basis for W_2.

$$g_2 = \dim W_2 = 1$$

$W_3 = $ Null Space of $(A+2I)$

$$A+2I = \begin{pmatrix} 3 & -3 & 3 \\ -2 & 2 & 2 \\ 1 & -1 & 5 \end{pmatrix}$$

$$(A+2I)x = \theta_3$$

$$W_3 = \left\{ \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : \gamma \in \mathbb{C} \right\}$$

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ basis for W_3

$\dim W_3 = 1$

$g_3 = 1$

Eigenvalues	AM	GM
4	1	1
2	1	1
-2	1	1

Ex 2: $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}$

$$C_A(\lambda) = (\lambda - 4)^2 (\lambda - 2)$$

$\lambda_1 = 4$ $\quad\quad$ $a_1 = 2$

$\lambda_2 = 2$ $\quad\quad$ $a_2 = 1$

$\underline{\underline{W_1}}$: Null Space of $A - 4I$

$$(A - 4I) = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(A - 4I)x = \theta_3$$

$$W_1 = \left\{ \begin{pmatrix} \alpha \\ \beta \\ \alpha + \beta \end{pmatrix} : \alpha, \beta \in \mathbb{C} \right\}$$

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is a basis for W_1

$\dim W_2 = 2 \qquad \therefore g_\beta = 2$

$\underline{\underline{W_2}}$: Null Space $A - \lambda_2 I$

$=$ Null Space $A - 2I$

$$A - 2I = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$(A - 2I)x = \theta_3$$

$$W_2 = \left\{ \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : \beta \in C \right\}$$

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ basis for W_2

$\dim W_2 = 1$ $\qquad$ $g_2 = 1$

Eigenvalue	AM	GM
4	2	2
2	1	1

Example

$$A = \begin{pmatrix} 2 & 0 & 2 \\ -1 & 3 & 1 \\ -1 & 1 & 5 \end{pmatrix}$$

$$C_A(\lambda) = |\lambda I - A|$$

$$= \begin{vmatrix} \lambda - 2 & 0 & -2 \\ 1 & \lambda - 3 & -1 \\ 1 & -1 & \lambda - 5 \end{vmatrix}$$

$$= (\lambda - 4)^2 (\lambda - 2)$$

$\lambda_1 = 4 \qquad a_1 = 2$

$\lambda_2 = 2 \qquad a_2 = 1$

$\underline{W_1}$: Null space $(A - \lambda_1 I)$

$= $ Null space $(A - 4I)$

$$A - 4I = \begin{pmatrix} -2 & 0 & 2 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$(A - 4I)x = \theta_3$$

$$W_1 = \left\{ \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is a basis for W_1

$\dim W_1 = 1 \qquad \therefore g_1 = 1$

$\underline{W_2}$: Null Space $(A - \lambda_2 I)$

$= \text{Null Space}(A - 2I)$

$$A - 2I = \begin{pmatrix} 0 & 0 & 2 \\ -1 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix}$$

$$(A - 2I)x = \theta_3$$

$$W_2 = \left\{ \beta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : \beta \in \mathbb{C} \right\}$$

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is a basis for W_2

$\therefore \dim W_2 = 1 \Rightarrow g_2 = 1$

Eigenvalue	AM	GM	
→ 4	2	1	
2	1	1	

Remark:

Suppose $\lambda_1, \dots, \lambda_r$ are some of the distinct eigenvalues of A

& $\varphi_1, \varphi_2, \dots, \varphi_r$ are corresp. eigenvectors

$\varphi_j \neq \theta_n$ & $A\varphi_j = \lambda_j \varphi_j$, $j = 1, 2, \dots, r$

We can show that

$\varphi_1, \varphi_2, \dots, \varphi_r$ are l.i

In short

Eigenvectors corresponding to distinct eigenvalues are l.i

SIMPLEST CASE

$A \in \mathbb{C}^{n \times n}$

$$C_A(\lambda) = (\lambda - \lambda_1)^{a_1} (\lambda - \lambda_2)^{a_2} \cdots (\lambda - \lambda_k)^{a_k}$$

W_j = Eigenspace corresponding to λ_j

= Null space of $(A - \lambda_j I)$

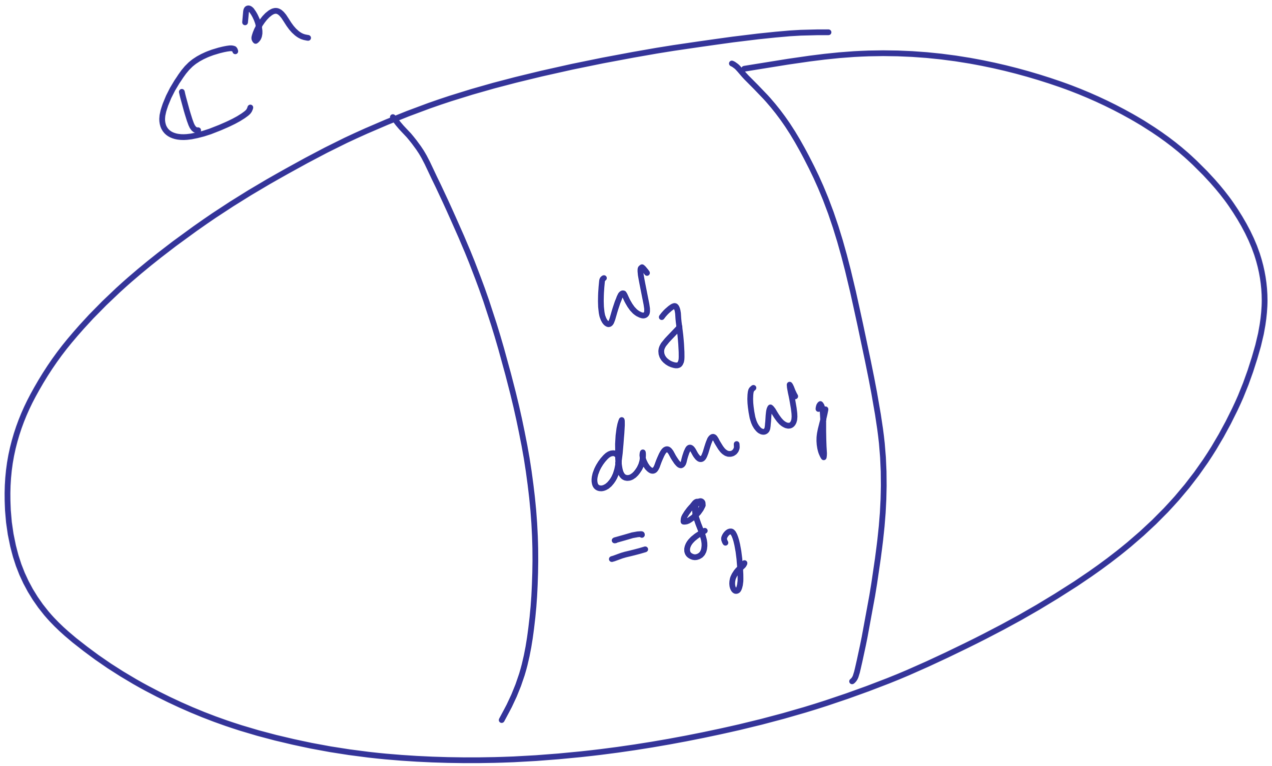
$g_j = \dim W_j$

We know $1 \le g_j \le a_j$

SUPPOSE $g_j = a_j$ for every eigenvalue λ_j.

We have therefore

$$g_1 + g_2 + \dots + g_k$$
$$= a_1 + a_2 + \dots + a_k$$
$$= n$$



$g_j = \dim W_j \implies a_j = \dim W_j$ ($\because$ we have assumed $a_j = g_j \ \forall j$)

This means

We can find a basis

consisting of (g_j vectors for W_j

$= a_j$)

say $\varphi_1^{(j)}, \varphi_2^{(j)}, \ldots, \varphi_{a_j}^{(j)}$

These are all eigenvectors corresp. to λ_j

Therefore $(\lambda_j, \underline{\varphi_1^{(j)}})$, $(\lambda_j, \underline{\varphi_2^{(j)}})$, $\ldots$, $(\lambda_j, \underline{\varphi_{a_j}^{(j)}})$

gives a_j eigenpairs for A

(Note $\varphi_1^{(j)}, \ldots, \varphi_{a_j}^{(j)}$ are l.i)

$W_1 \longrightarrow a_1$ eigenpairs $\longleftarrow$

$\updownarrow$?

$W_2 \longrightarrow a_2$ eigenpairs $\longleftarrow$

$\vdots$

$W_k \longrightarrow a_k$ eigenpairs $\longleftarrow$

If we can show that as claimed before that eigenvectors corresp to distinct eigenvalues are l.i

Then there will give us together

$$a_1 + a_2 + \cdots + a_k = n$$

eigenpairs in which all the

eigenvectors involved are l.i

Hence A will be diagonalizable.

CONCLUSION

$A \in \mathbb{C}^{n \times n}$

$GM = AM$ for every eigenvalue

$\Longrightarrow$ A has n eigenpairs in which all eigenvectors are l.i

$\Longrightarrow$ A is diagonalizable

PROVIDED the following holds:

$\boxed{\text{Eigenvectors Corresp. to Distinct Eigenvalues are l.i}}$

Very Special Case

$A \in \mathbb{C}^{n \times n}$ has n distinct eigenvalues.

$A = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$

a.m of $\lambda_j = 1$

$= g.m. \text{ of } \lambda_j$

We have A is diagonalizable