$A \in \mathbb{C}^{n \times n}$

$$c_A(\lambda) = \det(\lambda I - A)$$
$$= (\lambda - \lambda_1)^{a_1} (\lambda - \lambda_2)^{a_2} \cdots (\lambda - \lambda_k)^{a_k}$$

$\lambda_1, \dots, \lambda_k$ are distinct eigvalues of A

$a_1, \dots, a_k$ algebraic mult. of $\lambda_1, \dots, \lambda_k$

W_j (Eigenspace corr. to λ_j)

$=$ Null space of $(A - \lambda_j I)$

$g_j = \dim W_j$ geometric mult. of λ_j

We had the following result:

If $g_j = a_j$ for each eigenvalue λ_j
then A is diagonalizable

We got this assuming that

Eigenvectors corresp. to
distinct eigenvalues
are l.i.

Ex 1

$$A = \begin{pmatrix} 1 & -3 & 3 \\ -2 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix}$$

$$C_A(\lambda) = (\lambda - 4)(\lambda - 2)(\lambda + 2)$$

$\lambda_1 = 4$, $a_1 = 1$

$\lambda_2 = 2$, $a_2 = 1$

$\lambda_3 = -2$, $a_3 = 1$

We found

$$W_1 = \left\{ x \in \mathbb{C}^3 : x = \alpha \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

$$W_2 = \left\{ x \in \mathbb{C}^3 : x = \beta \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} : \beta \in \mathbb{C} \right\}$$

$$W_3 = \left\{ x \in \mathbb{C}^3 : x = \gamma \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \gamma \in \mathbb{R} \right\}$$

$g_1 = \dim W_1 = 1$

$g_2 = \dim W_2 = 1$

$g_3 = \dim W_3 = 1$

$$\left.\begin{aligned} a_1 &= g_1 = 1 \\ a_2 &= g_2 = 1 \\ a_3 &= g_3 = 1 \end{aligned}\right\} \Rightarrow \text{am} = \text{gm}$$

for each eigenvalue

Hence A is diagonalizable

Any eigvect of λ_1 is of the form

$$\begin{pmatrix} \alpha \\ 0 \\ \alpha \end{pmatrix} ; \quad \alpha \neq 0$$

Any eigvect of λ_2 is of the form

$$\begin{pmatrix} 0 \\ \beta \\ \beta \end{pmatrix} ; \quad \beta \neq 0$$

Any eigvect of λ_3 is of the form

$$\begin{pmatrix} \gamma \\ \gamma \\ 0 \end{pmatrix} , \quad \gamma \neq 0$$

These eigvect. which correspond to distinct eigenvalues can be easily verified to be l.i.

We shall now look at the process of proving that for $A \in \mathbb{C}^{n \times n}$ eigenvect corr. to distinct eigenvalues are l.i

Polynomials.

$$p(\lambda) = a_0 + a_1 \lambda + \cdots + a_r \lambda^r \qquad (a_r \neq 0)$$

where $a_0, a_1, \ldots, a_r \in \mathbb{C}$

$A \in \mathbb{C}^{n \times n}$

Define powers of A recursively as

$$A^0 = I_{n \times n}$$

$$A^1 = A$$

$$A^2 = A \times A$$

Having defined A^r we define

$$A^{r+1} = A A^r$$

(Easy to see $A^r A^s = A^s A^r = A^{r+s}$)

For any given polynomial

$$p(\lambda) = a_0 + a_1\lambda + \cdots + a_r\lambda^r$$

we define

$$p(A) = a_0 I_{n\times n} + a_1 A + a_2 A^2 + \cdots + a_n A^n$$

$$\in \mathbb{C}^{n\times n}$$

Some Special Polynomials

$A \in \mathbb{C}^{n \times n}$

Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct eigenvalues of A

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_k)$$

This is monic polynomial of degree k whose roots are the distinct eigenvalues of A

We now construct

$$\begin{cases} p_1(\lambda) = (\lambda-\lambda_2)(\lambda-\lambda_3)\cdots(\lambda-\lambda_k) \\ p_2(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_3)\cdots(\lambda-\lambda_k) \\ \vdots \\ p_k(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_{k-1}) \end{cases}$$

$$p_j(\lambda) = \frac{p(\lambda)}{(\lambda-\lambda_j)} = \underline{\prod_{\substack{r=1 \\ r\neq j}}^{k} (\lambda-\lambda_r)}$$

$j = 1, 2, 3, \ldots, k$

Each of these k polynomials is a monic poly of degree $(k-1)$

$$p_j(\lambda_r) = 0 \quad \text{if } r \neq j$$

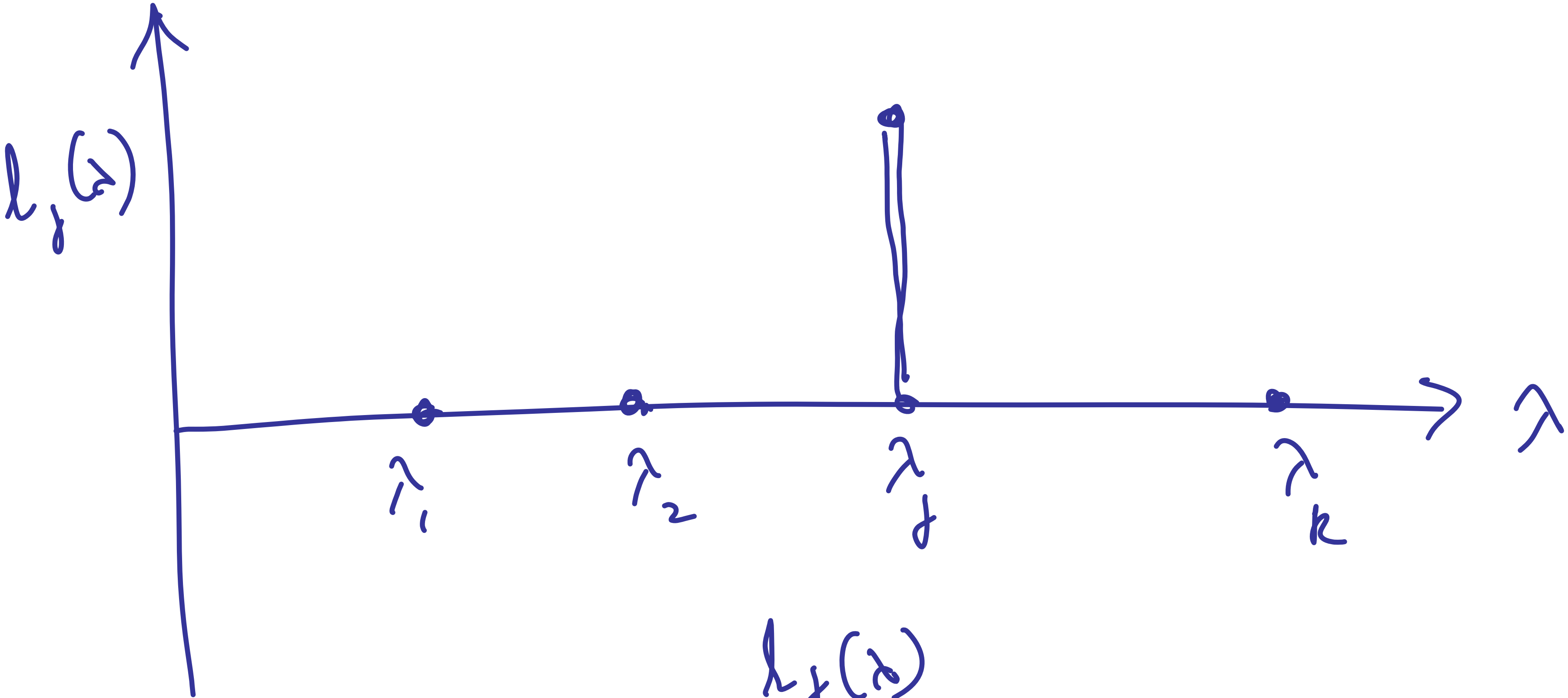
$$= \underbrace{\prod_{\substack{r=1 \\ r \neq j}}^{k} (\lambda_j - \lambda_r)}_{p_j(\lambda_j)} \quad \text{if } r = \lambda$$

$$l_j(\lambda) = \frac{p_j(\lambda)}{p_j(\lambda_j)} = \prod_{r=1}^{k} \frac{(\lambda - \lambda_r)}{(\lambda_j - \lambda_r)}$$

$$\left.\begin{aligned} l_j(\lambda_j) &= 1 \\ l_j(\lambda_r) &= 0 \end{aligned}\right\} \text{if } r \neq j$$

$$l_j(\lambda_r) = \begin{cases} 1 & \text{if } r = j \\ 0 & \text{if } r \neq j \end{cases}$$

$$\{ l_j(\lambda) \}_{j=1,2,\cdots,k}$$



$l_j(\lambda)$

$l_1(\lambda), l_2(\lambda), \ldots, l_k(\lambda)$

are called the Lagrange Interpolation polynomials corr. to the points

$$\lambda_1, \lambda_2, \ldots, \lambda_k$$

Examples: $A = \begin{pmatrix} 1 & -3 & 3 \\ -2 & 0 & 2 \\ 1 & -1 & 3 \end{pmatrix}$

$$C_A(\lambda) = (\lambda-4)(\lambda-2)(\lambda+2)$$

$\lambda_1 = 4 \;;\; \lambda_2 = 2 \;;\; \lambda_3 = -2$

Let us construct the Lagrange Interpol. poly corresponding to these $\lambda_1, \lambda_2, \lambda_3$

$$l_1(\lambda) = \frac{(\lambda - 2)\,(\lambda - (-2))}{(4-2)\,(4-(-2))} = \frac{(\lambda-2)(\lambda+2)}{2 \times 6} = \frac{\lambda^2 - 4}{12}$$

$$l_2(\lambda) = \frac{(\lambda - 4)\,(\lambda - (-2))}{(2-4)\,(2-(-2))} = \frac{(\lambda-4)(\lambda+2)}{-8}$$

$$= \frac{\lambda^2 - 2\lambda - 8}{-8}$$

$$l_3(\lambda) = \frac{(\lambda-4)(\lambda-2)}{(-2-4)(-2-2)} = \frac{\lambda^2-6\lambda+8}{24}$$

<u>Example:</u>

$$A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ 0 & 0 & 4 \end{pmatrix}$$

$$C_A(\lambda) = (\lambda-4)^2(\lambda-2)$$

$\lambda_1 = 4 \qquad \lambda_2 = 2$

<u>Lagrange Poly:</u>

$$l_1(\lambda) = \frac{(\lambda-2)}{(4-2)} = \frac{\lambda-2}{2}$$

$$l_2(\lambda) = \frac{(\lambda-4)}{(2-4)} = -\frac{\lambda-4}{2}$$

$A \in \mathbb{C}^{n\times n}$, $\lambda_1, \dots, \lambda_k$ distinct eigenvalues corresponding to the k Lagrange Interpolation polynomials $l_1(\lambda), l_2(\lambda), \dots, l_k(\lambda)$ we have $l_1(A), l_2(A), \dots, l_k(A)$

Suppose u is an eigenvect

corresp. to λ_j

That is

$$Au = \lambda_j u \quad (\text{and } u \neq \theta_n)$$

$$A(Au) = A(\lambda_j u)$$

$$A^2 u = \lambda_j (Au)$$

$$= \lambda_j^2 u$$

Similarly we get

$$A^r u = \lambda_j^r u \quad \text{for every nonneg integer } r$$

Suppose now

$$p(\lambda) = a_0 + a_1\lambda + \cdots + a_r\lambda^r$$

is any poly,

then

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_r A^r$$

$$\Rightarrow \quad p(A)u = (a_0 I + a_1 A + a_2 A^2 + \cdots + a_r A^r)u$$

$$= a_0 u + a_1(\lambda_j u) + a_2(\lambda_j^2 u) + \cdots + a_r \lambda_j^r u$$

$$= (a_0 + a_1\lambda_j + a_2\lambda_j^2 + \cdots + a_r\lambda_j^r)u$$

$$= p(\lambda_j)u$$

Conclusion:

If λ_j is an eigenvalue of A and u is an eigenvector corresponding to λ_j,

Then for any polynomial $p(\lambda)$,

$$\boxed{p(A)u = p(\lambda_j)u}$$

$A \in \mathbb{C}^{n \times n}$

$\lambda_1, \lambda_2, ----, \lambda_k$ are the distinct eigenvalues of A

Suppose the corresponding Lagrange Interpolation poly are

$l_1(\lambda), l_2(\lambda), ---, l_k(\lambda)$

$$l_j(\lambda_r) \begin{array}{ll} = 0 & \text{if } r \neq j \\ = 1 & \text{if } r = j \end{array} \Bigg\}$$

Suppose

$$\varphi_1, \varphi_2, \ldots, \varphi_k$$

are eigenvectors corresp to the distinct eigenvectors

$$\lambda_1, \lambda_2, \ldots, \lambda_k$$

respectively

$$A\varphi_j = \lambda_j \varphi_j \quad , \; j = 1, 2, \ldots, k$$

$$(\text{and } \underline{\underline{\varphi_j \neq \theta_n}})$$

Hence

$$\boxed{p(A)\,\varphi_j = p(\lambda_j)\,\varphi_j.}$$

Suppose

$$\alpha_1\varphi_1+\alpha_2\varphi_2+\cdots+\alpha_k\varphi_k=\theta_n$$

$$\Rightarrow\quad l_j(A)\left[\alpha_1\varphi_1+\alpha_2\varphi_2+\cdots+\alpha_k\varphi_k\right]=l_j(A)\theta_n$$

$$=\theta_n$$

$$\Rightarrow\quad \alpha_1 l_j(A)\varphi_1+\alpha_2 l_j(A)\varphi_2+\cdots+\alpha_k l_j(A)\varphi_k=\theta_n$$

$$\Rightarrow\quad \alpha_1 \underline{l_j(\lambda_1)}\varphi_1+\alpha_2 \underline{l_j(\lambda_2)}\varphi_2+\underbrace{\cdots}_{\alpha_j l_j(\lambda_j)\varphi_j}+\alpha_k \underline{l_j(\lambda_k)}\varphi_k=\theta_n$$

$$\Rightarrow\quad \alpha_j\varphi_j=\theta_n \qquad \left(\because\ l_j(\lambda_i)=0 \text{ if } i\neq j,\quad l_j(\lambda_j)=1\right)$$

$$\Rightarrow\quad \alpha_j=0 \quad \text{for each } j=1,2,\ldots,k$$

$\Rightarrow \varphi_1, \varphi_2, \dots, \varphi_k$ l.i

Conclusion:

Eigenvectors corresponding to distinct eigenvalues are l.i

In Summary

$A \in \mathbb{C}^{n \times n}$

$C_A(\lambda) = (\lambda-\lambda_1)^{a_1} \cdots (\lambda-\lambda_k)^{a_k}$

$\lambda_1, \ldots, \lambda_k$ distinct eigenvalues

$a_1, \ldots, a_k$ their algebraic mult.

W_j (eigenspace corr. to λ_j)

$=$ Null space $(A - \lambda_j I)$

$g_j = \dim W_j$ the g.m. of A

SUPPOSE

$g_j = a_j$ for every eigvalue λ_j

Then A is diagonalizable.

In such a case
How to get P s.t

$P^{-1}AP$ is a diagonal matrix ?

W_j has a basis consisting
of a_j vectors $(\because g_j = \dim W_j$
$\quad (=a_j))$

Let this bases be

$\varphi_1^{(j)}, \varphi_2^{(j)}, \ldots, \varphi_{a_j}^{(j)}$

We do this for $j = 1, 2, 3, \ldots, k$

Construct the matrix P as

$$P = \left[\varphi_1^{(1)} \;\; \varphi_2^{(1)} \; \cdots \; \varphi_{a_1}^{(1)} \;\; \varphi_1^{(2)} \cdots \varphi_{a_2}^{(2)} \; \cdots \; \varphi_1^{(k)} \;\; \varphi_2^{(k)} \cdots \varphi_{a_k}^{(k)} \right]$$

P is invertible since columns are l.i.

and

$$P^{-1}AP = \begin{pmatrix} \overbrace{\lambda_1 \cdots \lambda_1}^{a_1 \text{ times}} & & & \text{\Large 0} \\ & \overbrace{\lambda_2 \cdots \lambda_2}^{a_2 \text{ times}} & & \\ & & \ddots & \\ \text{\Large 0} & & & \overbrace{\lambda_k \cdots \lambda_k}^{a_k \text{ times}} \end{pmatrix}$$

Whenever

$gm = am$ for every eigenvalue

No Problem about diagonalizability

Unfortunately this does not take place always.

For example

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$C_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = \lambda^2$$

$\lambda_1 = 0 \qquad a_1 = 2$

$$W_1 = \text{Null sp } (A - \lambda_1 I)$$
$$= \text{Null sp } A$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$Ax = \theta_2 \implies x_2 = 0$$

$$\implies W_1 = \left\{ x = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} : \alpha \in \mathbb{C} \right\}$$

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a basis for W_1

$\therefore \dim W_1 = 1$

$\therefore \quad g_1 = 1$

Hence

$a_1 = 2 \quad , \quad g_1 = 1$

$\underline{\underline{g_1 < a_1}}$