$A \in \mathcal{H}_n$ (i.e A Hermitian)

$u, v \in \mathbb{C}^n$ then we define

$v \otimes u = u v^*$ This is an $n \times n$ matrix

and $(v \otimes u)^* = (u v^*)^*$

$= (v^*)^* u^*$

$= v u^*$

$u \otimes u = u u^*$

$(u \otimes u)^* = u u^* = u \otimes u$

$u \otimes u$ is in $\mathcal{H}_n$ (i.e. a Hermitian matrix for every $u \in \mathbb{C}^n$)

For example if

$$u = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$u \otimes u = u u^* = \begin{pmatrix} 1 \\ i \end{pmatrix} \begin{pmatrix} 1 & -i \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

which is a Hermitian matrix

$A \in \mathcal{H}_n$

$$C_A(\lambda) = (\lambda - \lambda_1)^{a_1} \cdots (\lambda - \lambda_k)^{a_k}$$

W_j = Null Space of $(A - \lambda_j I)$

$\dim W_j = g_j = a_j$, $j = 1, 2, \cdots, k$

$\mathcal{B}_j : \varphi_1^{(j)}, \varphi_2^{(j)}, \cdots, \varphi_{a_j}^{(j)}$ (o.n. basis)

Then $\bigcup_{j=1}^{k} \mathcal{B}_j$ is a basis for $\mathbb{C}^n$.

We have seen that A can be written as

$$\sum_{j=1}^{k} \boxed{\sum_{r=1}^{a_j} \lambda_j \left(\varphi_r^{(j)} \otimes \varphi_r^{(j)} \right)}$$

Note that $\varphi_r^{(j)} \otimes \varphi_r^{(j)}$ is an $n \times n$

Hermitian matrix of rank one.

In Particular, if 0 is an eigenvalue of multiplicity a_k (i.e. $\lambda_k = 0$)

Null Sp of $(A - 0I)$ = Null Sp of A has dim a_k

$\therefore \boxed{\nu_A = a_k}$

hence the ν_A terms corr. to $\lambda_k = 0$ disappear in the above sum and we get

$$A = \sum_{j=1}^{k-1} \sum_{r=1}^{a_j} \boxed{\underline{\lambda_j}\, \underline{\varphi_r^{(j)} \otimes \varphi_r^{(j)}}}$$

$(n - \nu_A$ terms i.e ρ_A terms$)$

Thus we have decomposed $A \in \mathcal{H}_n$ of rank ρ_A as the sum of ρ_A Hermitian matrices of rank one

If A is real symm we replace $*$ by T

Examples

(1) $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

Real Symm matrix

$C_A(\lambda) = (\lambda-2)^2(\lambda-8)$

It has two distinct eigenvalues

$\lambda_1 = 2$; $a_1 = 2$

$\lambda_2 = 8$, $a_2 = 1$

0 is not an eigenvalue

$\therefore$ Nullity is 0

Rank of $A = 3$

We shall express A as the sum of 3 Hermitian matrices each of rank one.

W_1 = Null space $(A - 2I)$

$$= \left\{ x = \begin{pmatrix} \alpha \\ \beta \\ -2\alpha + \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R} \right\}$$

$\mathcal{B}_1$: o.n. basis for W_1

$$\underbrace{\varphi_1^{(1)} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}}_{\lambda_1 = 2} \quad ; \quad \underbrace{\varphi_2^{(1)} = \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}}_{\lambda_1 = 2}$$

$$W_2 = \text{Null Space}(A - 8I)$$

$$= \left\{ x = \begin{pmatrix} 2r \\ -r \\ r \end{pmatrix} : r \in \mathbb{R} \right\}$$

$$\varphi_1^{(2)} = \underbrace{\frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}_{\lambda_2 = 8}$$

$$\varphi_1^{(1)} \otimes \varphi_1^{(1)} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 0 & -2 \end{pmatrix}$$

$\lambda_1 = \cancel{8}\, 2$

$$= \frac{1}{5} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$ ✓ Real Symm

$$\varphi_2^{(1)} \otimes \varphi_2^{(1)} = \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 2 & 5 & 1 \end{pmatrix}$$

$\lambda_1 = 2$

$$= \frac{1}{30} \begin{pmatrix} 4 & 10 & 2 \\ 10 & 25 & 5 \\ 2 & 5 & 1 \end{pmatrix}$$ Real Symm

$$\varphi_1^{(2)} \otimes \varphi_1^{(2)} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & -1 & 1 \end{pmatrix}$$

$\lambda_2 = 8$

$$= \frac{1}{6} \begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}$$

$$\underset{\overset{\downarrow}{2}}{\lambda_1}\ \varphi_1^{(1)} \otimes \varphi_1^{(1)} + \underset{\overset{\downarrow}{2}}{\lambda_1}\ \varphi_2^{(1)} \otimes \varphi_2^{(1)} + \underset{\overset{\downarrow}{8}}{\lambda_2}\ \varphi_1^{(2)} \otimes \varphi_1^{(2)}$$

$$= \underbrace{\begin{pmatrix} \frac{2}{5} & 0 & -2/5 \\ 0 & 0 & 0 \\ -\frac{4}{5} & 0 & 8/5 \end{pmatrix}}_{\lambda_1 \varphi_1^{(1)} \otimes \varphi_1^{(1)}} + \underbrace{\begin{pmatrix} 4/15 & 10/15 & 2/15 \\ 10/15 & 25/15 & 5/15 \\ 2/15 & 5/15 & 1/15 \end{pmatrix}}_{\lambda_1 \varphi_2^{(1)} \otimes \varphi_2^{(1)}}$$

$$+ \underbrace{\begin{pmatrix} 16/3 & -8/3 & 8/3 \\ -8/3 & 4/3 & -4/3 \\ 8/3 & -4/3 & 4/3 \end{pmatrix}}_{\lambda_2 \varphi_1^{(2)} \otimes \varphi_1^{(2)}}$$

Check
$=$ A

Thus we have A as the sum of 3 Hermitian matrices of rank one

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Ex 2

$$A = \begin{pmatrix} 5 & 10 & 0 \\ 10 & 25 & 5 \\ 0 & 5 & 5 \end{pmatrix}$$

$$C_A(\lambda) = (\lambda - 30)(\lambda - 5)\lambda$$

$\lambda_1 = 30 \qquad a_1 = 1$

$\lambda_2 = 5 \qquad a_2 = 1$

$\lambda_3 = 0 \qquad a_3 = 1$

A has nullity 1

$\therefore$ A has rank 2 (real symm)

We shall decompose this as the sum of TWO Hermitian matrices each of rank one.

W_1 : Null Space $(A - \lambda_1 I)$

" " $(A - 30 I)$

$$= \left\{ \begin{pmatrix} 2\alpha \\ 5\alpha \\ \alpha \end{pmatrix} : \alpha \in \mathbb{R} \right\}$$

$$\varphi_1^{(1)} = \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \quad \longleftarrow \quad \lambda_1 = 30$$

$$W_2 = \text{Null space}\,(A - 5I)$$

$$= \left\{ \begin{pmatrix} \beta \\ 0 \\ -2\beta \end{pmatrix} : \beta \in \mathbb{R} \right\}$$

$$\varphi_1^{(2)} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$$\varphi_1^{(1)} \otimes \varphi_1^{(1)} = \frac{1}{\sqrt{30}} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \frac{1}{\sqrt{30}} \begin{pmatrix} 2 & 5 & 1 \end{pmatrix}$$

$$= \frac{1}{30} \begin{pmatrix} 4 & 10 & 2 \\ 10 & 25 & 5 \\ 2 & 5 & 1 \end{pmatrix} \longleftarrow \lambda_1 = 30$$

$$\varphi_1^{(2)} \otimes \varphi_1^{(2)} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 0 & -2 \end{pmatrix}$$

$$= \frac{1}{5} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix} \longleftarrow \lambda_2 = 5$$

Decomposition

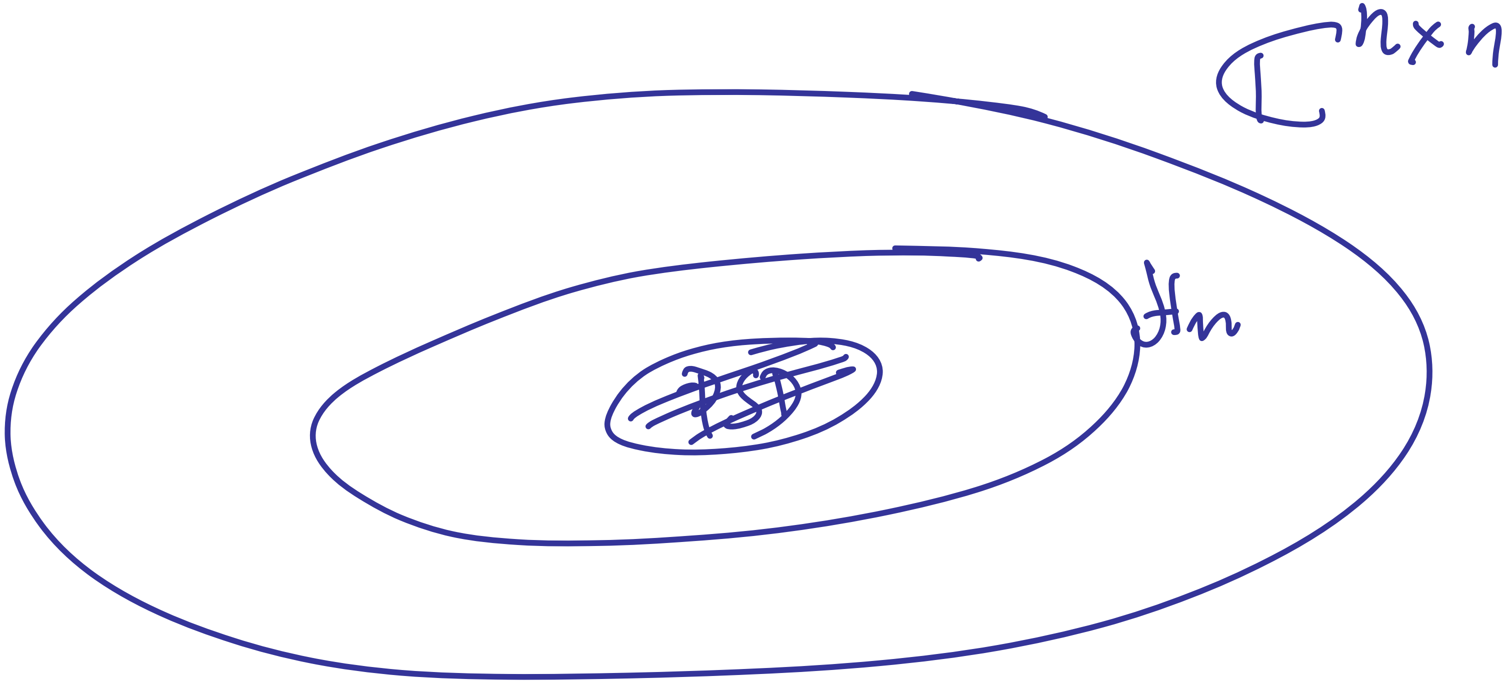
$$\underset{\downarrow \atop 30}{\lambda_1}\, \varphi_1^{(1)} \otimes \varphi_1^{(1)} + \underset{\downarrow \atop 5}{\lambda_2}\, \varphi_1^{(2)} \otimes \varphi_1^{(2)}$$

$$\begin{pmatrix} 4 & 10 & 2 \\ 10 & 25 & 5 \\ 2 & 5 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$$

Check A

(Sum of Two (Hermitian) Real Symm matrices each of rank one.

Special class of Hermitian Matrices



$A \in \mathcal{H}_n$ We know that

$$(Ax, x) \text{ is real } \forall x \in \mathbb{C}^n$$

However it will happen in general that

$$(Ax, x) > 0 \text{ for some } x \in \mathbb{C}^n$$
$$< 0 \text{ for some } x \in \mathbb{C}^n$$
$$= 0 \text{ for } x = \theta_n \text{ \& possibly for } x \neq \theta_n$$

We look at those $A \in \mathcal{H}_n$ for which

$$(Ax, x) \geq 0 \quad \forall\, x \in \mathbb{C}^n.$$

(Positive semidefinite matrices)

We know $(Ax, x) = 0$ when $x = \theta_n$

Suppose in addition to being Pos. Semi def we also have $(Ax, x) = 0$ only for $x = \theta_n$

i.e $\quad (Ax, x) > 0$ for $x \neq \theta_n$

we say A is Positive definite matrix

Definition

A Hermitian matrix $A \in \mathcal{H}_n$ is said to be Positive Semidefinite if
$$(Ax, x) \geq 0 \quad \forall x \in \mathbb{C}^n.$$

If further
$$(Ax, x) > 0 \quad \forall x \neq \theta_n, \; x \in \mathbb{C}^n$$

we say A is POSITIVE DEFINITE Matrix

Some Important Properties of PSD matrices

Let A be a Positive Semidefinite matrix

1) All Properties that we had for Hermitian matrices also hold for A

2) Suppose λ is an eigenvalue of A.

We know λ must be real

There exists a $\underline{u \neq \theta_n}$, $u \in \mathbb{C}^n$

s.t.

$$Au = \lambda u$$

$$\Rightarrow (Au, u) = (\lambda u, u)$$

$$= \lambda \underline{(u, u)} = \lambda \|u\|^2$$

> 0

$$\Rightarrow \lambda = \frac{(Au, u)}{\|u\|^2}$$

$((Au, u) \geq 0 \because A$ is PSD$)$

$$\Rightarrow \lambda \geq 0$$

<u>Conclusion:</u> All the eigenvalues of a PSD matrix must be real and nonnegative.

(3) Suppose A has nullity ν_A $\quad (\rho+\nu_A=n)$
Rank ρ

$\therefore$ $\lambda=0$ is an eigenvalue of multiplicity ν_A

All the remaining $n-\nu_A=\rho$ eigenvalues must be >0

$\therefore$ We can arrange the eigenvalues of a PSD matrix A as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p > 0 = \underbrace{\lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_n}_{\nu_A \text{ of these}}$$

(In particular if A is PD all eigenvalues are >0)

Corresponding to the 0 eigenvalue we can find

$$\varphi_1, \varphi_2, \ldots, \varphi_{\nu_A}$$

o.n. eigenvectors forming an o.n. basis for N_A

Corresponding to the Positive eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\rho > 0$ we get o.n. eigenvectors $v_1, v_2, \dots, v_\rho$

A is PSD

1) all eigenvalues are real ⟵
2) all eigenvalues are ≥ 0 ⟵
3) If nullity of A is ν_A & rank A is ρ then the n eigenvalues can be arranged as

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_p > 0 = \underbrace{\lambda_{p+1} = \cdots = \lambda_n}_{\nu_A \text{ of them}}$$

Get:

4) Corresponding o.n. eigenvectors

$$\underline{\underline{v_1, v_2, \cdots, v_p,}}\ \varphi_1, \varphi_2, \cdots, \varphi_{\nu_A}$$

<u>Note</u>:

$$\left.\begin{array}{l} Av_1 = \lambda_1 v_1 \\ Av_2 = \lambda_2 v_2 \\ \vdots \\ Av_p = \lambda_p v_p \end{array}\right\} \quad \lambda_1, \lambda_2, \ldots, \lambda_p > 0$$

We get
$$v_j = A\left(\frac{1}{\lambda_j} v_j\right) = A(x_j)$$

$$\Rightarrow \quad v_j \in \text{Range of } A \quad , \quad j = 1, \dots, p$$

$v_1, \dots, v_p \in$ Range A

$\therefore$ o.n. vectors in Range A

Since dim Range of $A = p$

& we have p o.n. vectors

$v_1, v_2, \dots, v_p$ in Range A

these form an o.n. basis for Range A

Conclusion

If A is PSD then the o.n. eigenvectors corr. to the POSITIVE eigenvalues provide an o.n. basis for Range of A