ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-10 INTEGRAL EQNS OF BL

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1. Development of Integral Equation for a Velocity BL
2. Development of Integral Equation for a Temperature BL

Integral Method - L10($\frac{1}{13}$)

1. The Integral Method represents a class of approximate methods capable of handling arbitrary variations of $U_\infty(x)$, $V_w(x)$ and $T_w(x)$
2. The method thus removes restrictions imposed by the Similarity Method
3. The method derives *exact* boundary layer equations in an *integral* form which are then solved in an approximate manner
4. The method is attractive because at least in the simple cases, *closed form* solutions can be obtained with little algebraic effort.

Velocity B L - L10($\frac{2}{13}$)

Consider Continuity and Momentum Equations

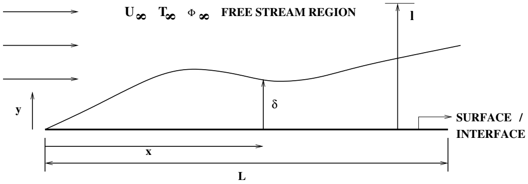
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$\frac{\partial (u\,u)}{\partial x} + \frac{\partial (v\,u)}{\partial y} = U_\infty \frac{d\,U_\infty}{d\,x} + \nu\,\frac{\partial^2 u}{\partial y^2} \tag{2}$$

The equations are integrated with respect to y
from y = 0 (u = 0, v = V_w)
to y = l (u = U_∞, v = V_l)

where l $> \delta_{max}$ in the region $0 < x < L$ (see next slide)

Integral Continuity Eqn - L10($\frac{3}{13}$)



$$
\begin{aligned}
\int_0^l \frac{\partial u}{\partial x}\, d\,y + \int_0^l \frac{\partial v}{\partial y}\, d\,y &= 0 \\
V_l - V_w = -\int_0^l \frac{\partial u}{\partial x}\, d\,y &= -\frac{\partial}{\partial x} \int_0^l u\, dy \qquad (3)
\end{aligned}
$$

V_l is a Ficticious velocity at y = l
$V_w(x)$ is the Suction/Blowing Velocity

Integral Momentum Eqn - 1 - L10($\frac{4}{13}$)

$$\int_0^l \frac{\partial(u\,u)}{\partial x}dy + \int_0^l \frac{\partial(v\,u)}{\partial y}dy = \int_0^l U_\infty \frac{d\,U_\infty}{d\,x}dy + \nu \int_0^l \frac{\partial^2 u}{\partial y^2}dy \quad (4)$$

$$\begin{aligned} \frac{d}{d\,x}\int_0^l u\,u\,d\,y + U_\infty\,V_l - u_w\,V_w \\ = \quad U_\infty \frac{d\,U_\infty}{d\,x}\,l + \nu \left\{ (\frac{\partial u}{\partial y})_{y=l} - (\frac{\partial u}{\partial y})_{y=0} \right\} \end{aligned}$$

Using *no-slip* condition $u_w = 0$ and noting that $\partial u/\partial y|_{y=l} = 0$

$$\frac{d}{d\,x}\int_0^l u u d y + U_\infty \left[V_w - \frac{d}{d\,x}\int_0^l u\,d\,y \right] = U_\infty \frac{d\,U_\infty}{d\,x}\,l - \frac{\tau_{w,x}}{\rho} \quad (5)$$

Integral Momentum Eqn - 2 - L10($\frac{5}{13}$)

Identity: $U_\infty \frac{d\,U_\infty}{d\,x}\, l = U_\infty \frac{d\,U_\infty}{d\,x} \int_0^l dy = \frac{d\,U_\infty}{d\,x} \int_0^l U_\infty dy$

Hence,

$$\frac{d}{d\,x}\int_0^l uu dy + U_\infty \left[V_w - \frac{d}{d\,x}\int_0^l u\,d\,y \right] = \frac{d\,U_\infty}{d\,x}\int_0^l U_\infty dy - \frac{\tau_{w,x}}{\rho}$$

Identity:

$$\frac{d}{d\,x}\int_0^l u\,U_\infty\,d\,y = \frac{d\,U_\infty}{d\,x}\int_0^l u\,d\,y + U_\infty \frac{d}{d\,x}\int_0^l u\,d\,y$$

Hence,

$$\frac{d}{d\,x}\int_0^l u\,(u - U_\infty)\,d\,y + \frac{d\,U_\infty}{d\,x}\int_0^l (u - U_\infty) dy = -\frac{\tau_{w,x}}{\rho} - U_\infty\,V_w$$

Integral Momentum Eqn- 3 - L10 ($\frac{6}{13}$)

Divide and Multiply by the same quantity

$$\frac{d}{d\,x}\left[U_\infty^2 \int_0^l \frac{u}{U_\infty}\,(\frac{u}{U_\infty} - 1)\,d\,y\right] + \; U_\infty \frac{d\,U_\infty}{d\,x} \int_0^l (\frac{u}{U_\infty} - 1)\,d\,y = -(\frac{\tau_{w,x}}{\rho} + V_w\,U_\infty)$$

Recall

1. $\delta_1 = \int_0^\infty (1 - \frac{u}{U_\infty})\,d\,y = \int_0^l (1 - \frac{u}{U_\infty})\,d\,y$
2. $\delta_2 = \int_0^\infty \frac{u}{U_\infty}\,(1 - \frac{u}{U_\infty})\,d\,y = \int_0^l \frac{u}{U_\infty}\,(1 - \frac{u}{U_\infty})\,d\,y$

Hence,

$$\frac{d}{d\,x}\left[U_\infty^2\,\delta_2\right] + U_\infty \frac{d\,U_\infty}{d\,x}\,\delta_1 = (\frac{\tau_{w,x}}{\rho} + V_w\,U_\infty)$$

Integral Momentum Eqn- 4 - L10 ($\frac{7}{13}$)

Dividing throughout by U_∞^2

$$\frac{d\,\delta_2}{d\,x} + \frac{1}{U_\infty}\frac{d\,U_\infty}{d\,x}\left(2\,\delta_2 + \delta_1\right) = \frac{C_{f,x}}{2} + \frac{V_w}{U_\infty} \qquad (6)$$

1. This is known as Integral Momentum Eqn
2. It is an Exact Equation - No assumptions are introduced
3. It is an ODE. Thus, PDEs of the BL are converted to an ODE for integral parameter δ_2
4. $C_{f,x} = \tau_{w,x}/(\rho\, U_\infty^2/2)$

Integral Kinetic Energy Eqn-1 - L10($\frac{8}{13}$)

$$\rho \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = \rho U_\infty \frac{d\, U_\infty}{d\, x} + \mu \frac{\partial^2 u}{\partial y^2} \qquad (7)$$

Multiply by u / ρ throughout

$$\frac{\partial u\, E}{\partial x} + \frac{\partial v\, E}{\partial y} = u\, U_\infty \frac{d\, U_\infty}{d\, x} + \nu\, u \frac{\partial^2 u}{\partial y^2} \qquad E = \frac{u^2}{2} \qquad (8)$$

Integrate from y = 0 to y = δ and note that $(vE)_{y=0} = 0$

$$\begin{aligned} \frac{d}{dx} \left[\int_0^\delta \left(\frac{u^3}{2}\right) dy \right] &+ \left[V_w - \frac{\partial}{\partial x} \int_0^\delta u\, dy \right] \frac{U_\infty^2}{2} \\ &= \frac{d\, U_\infty}{dx} \int_0^\delta u\, U_\infty\, dy \\ &+ \nu \int_0^\delta u \left(\frac{\partial^2 u}{\partial y^2}\right) dy \qquad (9) \end{aligned}$$

Integral Kinetic Energy Eqn-2 - L10($\frac{9}{13}$)

Integration by parts gives

$$\nu \int_0^{\delta} u \, (\frac{\partial^2 u}{\partial y^2}) \, dy = - \, \nu \int_0^{\delta} (\frac{\partial u}{\partial y})^2 \, dy$$

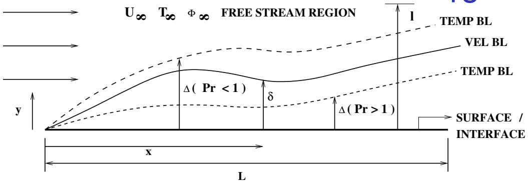
Define *Kinetic Energy Thickness* δ_3 as

$$\delta_3 \equiv \int_0^{\infty} \frac{u}{U_\infty} \left[1 - (\frac{u}{U_\infty})^2 \right] \, d \, y \qquad (10)$$

Manipulation gives Integral Kinetic Energy Eqn

$$\frac{d}{dx} \, (U_\infty^3 \, \delta_3) = V_w \, U_\infty^2 + 2 \, \nu \int_0^{\delta} (\frac{\partial u}{\partial y})^2 \, dy \qquad (11)$$

Integral Energy Eqn - 1 - L10($\frac{10}{13}$)



$$\frac{\partial(u\,T)}{\partial x} + \frac{\partial(v\,T)}{\partial y} = \alpha\,\frac{\partial^2 T}{\partial y^2} + \frac{\nu}{C_p}\,(\frac{\partial u}{\partial y})^2 \qquad (12)$$

Define $\theta = (T - T_\infty)/(T_w - T_\infty)$ $\quad T_\infty$ = constant, T_w = F (x)

$$\frac{\partial(u\,\theta)}{\partial x} + \frac{\partial(v\,\theta)}{\partial y} + \frac{u\,\theta}{(T_w - T_\infty)}\,\frac{d}{dx}(T_w - T_\infty)$$
$$= \alpha\,\frac{\partial^2\theta}{\partial y^2} + \frac{\nu}{C_p(T_w - T_\infty)}\,(\frac{\partial u}{\partial y})^2 \qquad (13)$$

The equation is integrated with respect to y from y = 0 to y = l where $l > \delta_{max}$ for Pr > 1 and $l > \Delta_{max}$ for Pr < 1

Integral Energy Eqn - 2 - L10($\frac{11}{13}$)

Integration gives

$$\frac{d}{dx}\left[U_\infty \int_0^l \frac{u}{U_\infty}\,\theta\,dy\right] + V_l\theta_l - V_w\theta_w$$

$$+ \quad \frac{1}{(T_w - T_\infty)}\,\frac{d}{dx}(T_w - T_\infty)\,\left\{U_\infty \int_0^l \frac{u}{U_\infty}\,\theta\,dy\right\}$$

$$= \quad \alpha\left\{(\frac{\partial\theta}{\partial y})_{y=l} - (\frac{\partial\theta}{\partial y})_{y=0}\right\} + \frac{\nu}{C_p(T_w - T_\infty)}\int_0^l (\frac{\partial u}{\partial y})^2\,dy$$

1. Recall $\Delta_2 = \int_0^\infty \frac{u\,(T - T_\infty)}{U_\infty (T_w - T_\infty)}\,dy = \int_0^l \frac{u}{U_\infty}\,\theta\,dy$
2. $\theta_l = \theta_\infty = 0$ and $\theta_w = 1$
3. $\alpha\,\frac{\partial\theta}{\partial y}|_{y=l} = 0$ and $\alpha\,\frac{\partial\theta}{\partial y}|_{y=0} = -\frac{q_w}{\rho\,Cp\,(T_w - T_\infty)} = -\frac{h_x}{\rho\,Cp}$

Integral Energy Eqn - 3 - L10($\frac{12}{13}$)

Substitution gives

$$\frac{d}{dx}\,[U_\infty \Delta_2] \;+\; \frac{U_\infty\,\Delta_2}{(T_w - T_\infty)}\,\frac{d}{dx}(T_w - T_\infty) = \frac{h_x}{\rho\,Cp} + V_w \\ +\; \frac{\nu}{C_p}\int_0^l \left(\frac{\partial u}{\partial y}\right)^2 dy \qquad (14)$$

Division by U_∞ gives Integral Energy Eqn

$$\frac{d\,\Delta_2}{d\,x} + \Delta_2\left[\frac{1}{(T_w - T_\infty)}\,\frac{d}{d\,x}(T_w - T_\infty) + \frac{1}{U_\infty}\frac{d\,U_\infty}{d\,x}\right] \\ = \; St_x + \frac{V_w}{U_\infty} + 2\,Ec_x\,\frac{\nu}{U_\infty^3}\int_0^l \left(\frac{\partial u}{\partial y}\right)^2 d\,y \qquad (15)$$

This ODE is Exact , $St_x = h_x/(\rho\,Cp\,U_\infty) = Nu_x/(Re_x\,Pr)$

Summary of Integral Eqns - L10($\frac{13}{13}$)

Continuity eqn

$$V_l - V_w = -\frac{\partial}{\partial x} \int_0^l u\, dy \tag{16}$$

Momentum Eqn

$$\frac{d\,\delta_2}{d\,x} + \frac{1}{U_\infty}\frac{d\,U_\infty}{d\,x}(2\,\delta_2 + \delta_1) = \frac{C_{f,x}}{2} + \frac{V_w}{U_\infty} \tag{17}$$

Energy Equation

$$\frac{d\,\Delta_2}{d\,x} + \Delta_2\left[\frac{1}{(T_w - T_\infty)}\frac{d}{d\,x}(T_w - T_\infty) + \frac{1}{U_\infty}\frac{d\,U_\infty}{d\,x}\right]$$
$$= \quad St_x + \frac{V_w}{U_\infty} + 2\,Ec_x\,\frac{\nu}{U_\infty^3}\int_0^l (\frac{\partial u}{\partial y})^2\, d\,y \tag{18}$$