ME-662 CONVECTIVE HEAT AND MASS TRANSFER

A. W. Date
Mechanical Engineering Department
Indian Institute of Technology, Bombay
Mumbai - 400076
India

LECTURE-12 INTEGRAL SOLNS TO LAMINAR TEMP BL

LECTURE-12 INTEGRAL SOLNS TO LAMINAR TEMP BL

$$\frac{d\,\Delta_2}{d\,x} + \Delta_2\left[\frac{1}{(T_w - T_\infty)}\frac{d}{d\,x}(T_w - T_\infty) + \frac{1}{U_\infty}\frac{d\,U_\infty}{d\,x}\right]$$
$$= \quad St_x + \frac{V_w}{U_\infty} + 2\,Ec_x\,\frac{\nu}{U_\infty^3}\int_0^l (\frac{\partial u}{\partial y})^2\,d\,y \qquad (1)$$

1. Solution Procedure
2. Solutions with Effects of Pressure Gradient and Sucrion/Blowing
3. Application to Flow over a Cylinder

Assumed Temp Profile - L12($\frac{1}{14}$)

$$\text{Let } T = a + b\,\eta_T + c\,\eta_T^2 + d\,\eta_T^3 + e\,\eta_T^4 \quad \eta_T = \frac{y}{\Delta} \tag{2}$$

At y = 0 (Wall)

$$T = T_w \tag{3}$$

$$\alpha \frac{\partial^2 T}{\partial y^2} = -\frac{\nu}{Cp}\left(\frac{\partial u}{\partial y}\right)^2 + V_w \frac{\partial T}{\partial y} \tag{4}$$

At y = Δ (Edge of BL)

$$T = T_\infty \tag{5}$$

$$\frac{\partial T}{\partial y} = 0 \tag{6}$$

$$\frac{\partial^2 T}{\partial y^2} = 0 \tag{7}$$

2nd BC derived from PDE

3rd BC ensures assymptotic behaviour as $y \rightarrow \delta$

Five BCs give 5 coefficients a, b, c, d and e

Derived Temp Profile - L12($\frac{2}{14}$)

$$\frac{T - T_\infty}{T_w - T_\infty} = 1 - 2\eta_T + 2\eta_T^3 - \eta_T^4 + A\left(\eta_T - 3\eta_T^2 + 3\eta_T^3 - \eta_T^4\right) \quad (8)$$

$$A = \frac{V_w^*\,(\Delta/\delta) + Ec\,(\Delta/\delta)^2\,(\lambda + 12)^2/(V_w^* + 6)^2}{3/Pr + V_w^*\,(\Delta/\delta)/2} \quad (9)$$

$$\frac{u}{U_\infty} = \left(\frac{6}{6 + V_w^*}\right)\left(F_1 + V_w^*\,F_2 + \lambda\,F_3\right) \quad V_w^* = \frac{V_w\,\delta}{\nu} \quad (10)$$

$$F_1 = 2\,\eta - 2\,\eta^3 + \eta^4 \quad F_2 = \frac{1}{6}(6\,\eta^2 - 8\,\eta^3 + 3\,\eta^4) \quad (11)$$

$$F_3 = \frac{1}{6}(\eta - 3\,\eta^2 + 3\,\eta^3 - \eta^4) \quad \lambda = \frac{\delta^2}{\nu}\,\frac{d\,U_\infty}{d\,x} \quad (12)$$

Evaluation of Δ_2 - L12($\frac{3}{14}$)

To make further progress, we need to evaluate Δ_2

$$\Delta_2 = \int_0^l \frac{u}{U_\infty} \left(\frac{T - T_\infty}{T_w - T_\infty}\right) dy \tag{13}$$

where l = Δ or δ which ever is greater.

1. This evaluation becomes extermely laborious
2. Hence, usually simplifications are made
3. For liquid metals, Pr $<< 1$, $(u/U_\infty) = 1$. Also, $\Delta >> \delta$ and hence, A $\rightarrow$ 2.
4. For liquids, V_w^* = 0 (not of interest). Hence,

$$A \rightarrow \left(\frac{Pr\,Ec}{3}\right) \left(\frac{\lambda + 12}{6}\right)^2 \left(\frac{\Delta}{\delta}\right)^2$$

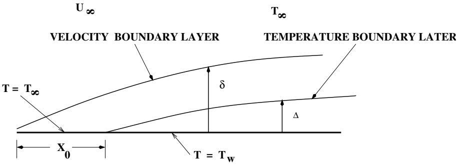
5. For Oils, Pr $>> 1$, $\Delta << \delta$. Hence, A $\rightarrow$ 0

Simple Case - L12($\frac{4}{14}$)

1. Consider a simple case of a Flat-Plate Boundary Layer with $V_w = 0$, U_∞ = const, Ec = 0
2. Pr > 1. Hence, $\delta > \Delta$
   T_w = Const for $x > x_0$
3. Then, the governing eqns are:

$$\frac{d\,\delta_2}{d\,x} = \frac{C_{f,x}}{2} = \frac{\tau_w}{\rho\, U_\infty^2}$$

$$\frac{d\,\Delta_2}{d\,x} = St_x = \frac{h_x}{\rho\, Cp\, U_\infty}$$



Assume simple profiles:

$$\frac{u}{U_\infty} = \frac{3}{2}\,\eta - \frac{1}{2}\,\eta^3$$

$$\frac{T - T_w}{T_\infty - T_w} = \frac{3}{2}\,\eta_T - \frac{1}{2}\,\eta_T^3$$

X_0 is *Unheated Starting Length*

Flat Plate Vel Solns L12($\frac{5}{14}$)

$$\frac{\delta_2}{\delta} = \frac{39}{280} \quad C_{f,x} = \frac{3}{2}\,\frac{\nu}{\delta\, U_\infty}$$

Substitution in Mom Eqn gives

$$\delta\,\frac{d\,\delta}{d\,x} = \frac{140}{13}\,\frac{\nu}{U_\infty}$$

Integration gives ($\delta_{x=0} = 0$)

$$\delta = \sqrt{\frac{280}{13}\,\frac{\nu\, x}{U_\infty}} = 4.64\,\sqrt{\frac{\nu\, x}{U_\infty}}$$

$$C_{f,x} = 0.646\, Re_x^{-0.5}$$

Exact Similarty Soln: $C_{f,x} = 0.664\, Re_x^{-0.5}$

Flat Plate Soln Pr > 1 L12($\frac{6}{14}$)

$$\frac{\Delta_2}{\Delta} = \frac{3}{20}\,R - \frac{3}{280}\,R^3 \quad St_x = \frac{3}{2}\,\frac{\alpha}{\Delta\,U_\infty} \quad R = \frac{\Delta}{\delta} < 1$$

Substitution in Energy Eqn

$$\begin{aligned} \frac{d\,\Delta_2}{d\,x} &= \frac{3\,\delta}{10}\left[R - \frac{R^3}{7}\right]\frac{d\,R}{d\,x} + \frac{3}{20}\left[R^2 - \frac{R^4}{14}\right]\frac{d\,\delta}{d\,x} = St_x \\ &\simeq \frac{3\,\delta\,R}{10}\,\frac{d\,R}{d\,x} + \frac{3\,R^2}{20}\,\frac{d\,\delta}{d\,x} = \frac{3}{2}\,\frac{\alpha}{\Delta\,U_\infty} \end{aligned}$$

Substituting for δ and $d\,\delta/dx$ gives

$$R^3 + 4\,R^2\,x\,\frac{d\,R}{d\,x} = \frac{13}{14\,Pr} \quad \text{or} \quad \frac{4}{3}\,x^{0.25}\,\frac{d}{d\,x}(x^{0.75}\,R^3) = \frac{13}{14\,Pr}$$

Flat Plate Soln Pr > 1 - Contd L12($\frac{7}{14}$)

Integrating and noting that $R = 0$ at $x = x_0$,

$$R^3 = (\frac{\Delta}{\delta})^3 = \frac{13}{14\, Pr} \left[1 - (\frac{x_0}{x})^{0.75} \right]$$

Therefore,

$$St_x = \frac{3\alpha}{2\Delta\, U_\infty} = \frac{3\, \alpha}{2\, R\, \delta\, U_\infty} = 0.331\, Re_x^{-0.5} Pr^{-0.66} \left[1 - (\frac{x_0}{x})^{0.75} \right]^{-0.33}$$

For $X_0 = 0$, Similarity Soln: $St_x = 0.33\, Re_x^{-0.5}\, Pr^{-0.66}$

We shall make use of this equation in a later development called Superposition Theory (see next lecture)

Effect of Pr Gr - L12($\frac{8}{14}$)

V_w = Ec = 0, T_w = const

$$\frac{d\,\Delta_2}{d\,x} + \frac{\Delta_2}{U_\infty}\frac{d\,U_\infty}{d\,x} = St_x = \frac{h_x}{\rho\,Cp\,U_\infty}$$

Define Conduction Thickness $\Delta_4 \equiv k/h_x \propto \Delta_2$. Hence, $St_x = \alpha/(\Delta_4\,U_\infty)$

Like momentum thickness δ_2, Postulate[1] a relationship

$$\frac{d\,\Delta_4}{d\,x} = F\left(U_\infty, \frac{d\,U_\infty}{d\,x}, \nu, \Delta_4, \mathrm{Pr}\right)$$

[1] Eckert E. R. G. and Weise W. *Messung der Temperaturverteilung auf der Oberfläche schnell strömter Kröper*, Forschg. Ing.-Wes., vol. 13, p 246 - 254, 1942

Dimensional Analysis - L12 ($\frac{9}{14}$)

1. let X, Y and Z represent characteristic length dimensions in x, y and z directions.Then, each parameter has following dimensions:

$$(\Delta_4 \rightarrow Y), (U_\infty \rightarrow \frac{X}{t}), (\frac{d\,U_\infty}{d\,x} \rightarrow \frac{1}{t})\,(\nu \rightarrow \frac{Y^2}{t}), (\frac{d\,\Delta_4}{d\,x} \rightarrow \frac{Y}{X})$$

2. Hence, for a fixed Prandtl number,

$$\frac{Y}{X} = (\frac{X}{t})^a\,(\frac{1}{t})^b\,(\frac{Y^2}{t})^c\,Y^d$$

3. Equating the like exponents, it is easy to show that:

$$\frac{U_\infty}{\nu}\frac{d\,\Delta_4^2}{d\,x} = F\,(\frac{\Delta_4^2}{\nu}\,\frac{d\,U_\infty}{d\,x}) = F(\kappa_T) \qquad (14)$$

Determination of Functional - L12 ($\frac{10}{14}$)

1. $U_\infty = Cx^m$ is a special case of arbitrary variation of U_∞(x). Hence, the functional must admit similarity *wedge flow* solutions. Therefore, with $Nu_x\, Re_x^{-0.5} = -\theta'\,(0) = C_1(m)$

$$\frac{U_\infty}{\nu}\frac{d\,\Delta_4^2}{d\,x} = \frac{1-m}{C_1^2} \quad \frac{\Delta_4^2}{\nu}\,\frac{d\,U_\infty}{d\,x} = \frac{m}{C_1^2} = \kappa_T$$

2. Hence, for a fixed Prandtl number

$$\frac{1-m}{C_1^2} = F\,(\,\frac{m}{C_1^2}) = F\,(\kappa_T)$$

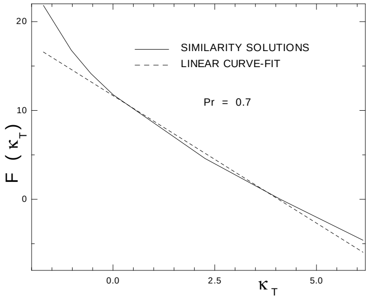
F (κ_T) vs κ_T ($V_w = 0$) - L12($\frac{11}{14}$)

From known similarity solutions for Pr = 0.7, the relationship is nearly linear.
Y-intercept - Flat Plate
X-intercept - Stagnation
Using C_1(m = 0) = 0.293 and C_1(m = 1) = 0.493

$$\frac{U_\infty}{\nu}\frac{d\,\Delta_4^2}{d\,x} = 11.67 - 2.87\frac{\Delta_4^2}{\nu}\frac{d\,U_\infty}{d\,x}$$

Further manipulation gives

$$\Delta_4^2 = \frac{11.67\,\nu}{U_\infty^{2.87}}\int_0^x U_\infty^{1.87}\,d\,x$$



Closed Form Soln $V_w^* = 0$ - L12($\frac{12}{14}$)

$$St_x = \frac{\alpha}{U_\infty \, \Delta_4} = 0.418 \, \nu^{0.5} \, U_\infty^{0.435} \left[\int_0^x U_\infty^{1.87} \, d\, x \right]^{-0.5} \quad Pr = 0.7$$

In general

$$St_x = K_1 \, \nu^{0.5} \, U_\infty^{K_2} \left[\int_0^x U_\infty^{K_3} \, d\, x \right]^{-0.5}$$

where K_1, K_2 and, K_3 are functions of Prandtl number.

Flow over a Cylinder L12($\frac{13}{14}$)

For flow over an impervious cylinder, with $x^* = x/D$

$$\frac{U_\infty}{V_a} = 2\sin(2\,x^*) = F(x^*)$$

Then, for Pr = 0.7 and T_w = const

$$\frac{\Delta_4}{D}\,Re_D^{0.5} = \frac{3.416}{F^{1.435}}\left[\int_0^{x^*} F^{1.87}\,dx^*\right]^{0.5}$$

and

$$St_x\,Re_D^{0.5} = \frac{h_x}{\rho\,Cp\,V_a}\,Re_D^{0.5} = \frac{0.418\,F^{0.435}}{\left[\int_0^{x^*} F^{1.87}\,dx^*\right]^{0.5}}$$

Evaluation:

$$\overline{St}_{sep}\,Re_D^{0.5} = \frac{1}{x_{sep}}\int_0^{x_{sep}} St_x\,Re_D^{0.5}\,dx = 2.686$$

Angular Variations Pr = 0.7 L12($\frac{14}{14}$)

θ deg	$(\Delta_4/D)Re_D^{0.5}$	$St_x Re_D^{0.5}$
0.0573	0.242E+01	0.296E+03
0.515	0.117E+01	0.679E+02
2.00	0.105E+01	0.194E+02
4.98	0.103E+01	0.801E+01
10.0	0.102E+01	0.402E+01
30.0	0.105E+01	0.136E+01
50.0	0.113E+01	0.821E+00
70.0	0.128E+01	0.592E+00
80.0	0.139E+01	0.521E+00
90.0	0.153E+01	0.465E+00
100.0	0.173E+01	0.419E+00
105.0	0.184E+01	0.401E+00
108.3	0.194E+01	0.388E+00