ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-18 FULLY-DEVELOPED LAMINAR FLOW HEAT TRANSFER-2

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Nusselt number - Ducts of Arbitrary Cross-Section

1. For Rectangular Duct family, Fourier Series solutions can be obtained.Same for Annular sector family
2. Here, the general method for arbitrary cross-sections introduced in lecture 16 is extended to heat transfer.
3. This method can be used for arbitrary circumferential variations of the thermal boundary conditions q_w, T_w and h_w

Problem Definition - 1 - L18($\frac{1}{21}$)

Governing Eqn (velocity)

$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{d\,p}{d\,x} = \text{Const}$$

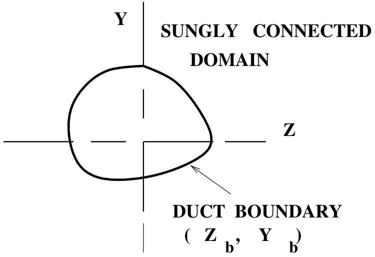
Define

$$\frac{u}{-\frac{1}{\mu}\frac{d\,p}{d\,x}} = u^* - (\frac{z^2+y^2}{4})$$

Hence, Laplace Eqn

$$\frac{\partial^2 u^*}{\partial z^2} + \frac{\partial^2 u^*}{\partial y^2} = 0$$

$$\text{with } u_b^* = (\frac{z_b^2+y_b^2}{4})$$



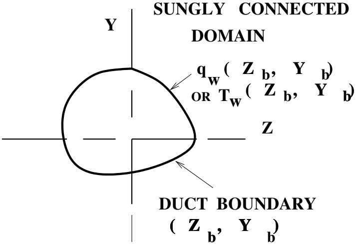
Soln

$$u^* = \sum_{i=1}^{N} c_{u_i} \times g_i$$

where c_{u_i} depend on boundary shape and g_i are N functions of z and y (see lecture 16)

Problem Definition - 2 - L18($\frac{2}{21}$)

Governing Eqn (Temparature)



$$\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial y^2} = \frac{u}{\alpha}\,\frac{dT_b}{dx}$$

$$\frac{d\,T_b}{dx} = \text{const} = \frac{\overline{q}_w\,D_h}{4\,\rho\,c_p\,\overline{u}}$$

Substitute $\overline{u} = 0.5\,(-1/\mu)\,(dp/dx)\,D_h^2/(f\,Re)$. Hence,

$$\frac{\partial^2 T}{\partial z^2} + \frac{\partial^2 T}{\partial y^2} = (\frac{8\,f\,Re\,\overline{q}_w}{k\,D_h^3}) \times \frac{u}{-(1/\mu)\,(d\,p/d\,x)}$$

$$= (\frac{8\,f\,Re\,\overline{q}_w}{k\,D_h^3}) \times \left\{u^* - (\frac{z^2+y^2}{4})\right\}$$

Further Development - 1 - L18($\frac{3}{21}$)

$$\text{Define} \quad \theta \quad = \quad T\,(\frac{8\,f\,Re\,\overline{q}_w}{k\,D_h^3})^{-1} \quad .\text{Then}$$

$$\frac{\partial^2\theta}{\partial z^2} \quad + \quad \frac{\partial^2\theta}{\partial y^2} = \sum_{i=1}^{N} c_{u_i}\, g_i - \quad (\frac{z^2+y^2}{4})$$

Now, let

$$\theta = \theta^* + \sum_{i=1}^{N} c_{u_i} \times G_i - (\frac{z^4+y^4}{48})$$

Then

$$\frac{\partial^2\theta}{\partial z^2} + \frac{\partial^2\theta}{\partial y^2} = \frac{\partial^2\theta^*}{\partial z^2} + \frac{\partial^2\theta^*}{\partial y^2} + \sum_{i=1}^{N} c_{u_i}\,(\frac{\partial^2 G_i}{\partial z^2} + \frac{\partial^2 G_i}{\partial y^2}) - (\frac{z^2+y^2}{4})$$

Further Development - 2 - L18($\frac{4}{21}$)

Now, if

$$(\frac{\partial^2 G_i}{\partial z^2} + \frac{\partial^2 G_i}{\partial y^2}) = g_i \quad \text{(solns on next slide)}$$

Then, it follows that

$$\begin{aligned} \frac{\partial^2 \theta^*}{\partial z^2} &+ \frac{\partial^2 \theta^*}{\partial y^2} = 0 \\ \text{Soln} \quad \theta^* &= \sum_{i=1}^{N} a_i \times g_i \quad \text{(as per velocity problem)} \\ \text{and} \quad \theta &= \sum_{i=1}^{N} (a_i \times g_i + c_{u_i} \times G_i) - (\frac{z^4 + y^4}{48}) \end{aligned}$$

where $a_i = c_{tw,i}$, $c_{qw,i}$ or $c_{hw,i}$ are functions of boundary conditions.

G_i Functions - L18($\frac{5}{21}$)

$$G_1 = 0.25\,(z^2 + y^2)$$
$$G_2 = (z^3 + 3\,y^2 z)/12$$
$$G_3 = (3z^2 y + y^3)/12$$
$$G_4 = (z^4 - y^4)/12$$
$$G_5 = (z^3 y + y^3 z)/6$$
$$G_6 = (z^5 - 5zy^4)/20$$
$$G_7 = (5z^4 y - y^5)/20$$
$$G_8 = (z^6 + y^6)/20 - (z^4\,y^2 + z^2 y^4)/4$$
$$G_9 = (z^5\,y - z\,y^5)/5$$
$$G_{10} = z^7/14 - z^5\,y^2 + 5\,z^3\,y^4/6$$
$$G_{11} = (yz^6 - z^2 y^5)/4 + y^7/28 - 5\,y^3\,z^4/12$$
$$G_{12} = (z^8 - y^8)/28 - (z^6\,y^2 - z^2\,y^6)/2$$
$$G_{13} = 3(z^7 y + zy^7)/14 - (z^5\,y^3 + z^3 y^5)/2$$
$$G_{14} = z^9/18 - 1.5\,z^7\,y^2 + 3.5\,z^5\,y^4 - 7\,y^6\,z^3/6$$
$$G_{15} = x^8 y/8 - 1.75\,y^5\,z^4 + y^7\,z^2 - y^9/24$$
$$G_{16} = (z^{10} + y^{10})/36 - 7(z^6\,y^4 + z^4 y^6)/6 - 0.75(z^8\,y^2 + z^2\,y^8)$$
$$G_{17} = 4z^9 y/9 - 4\,z^7\,y^3 + 28z^5 y^5/5 - 4\,z^3\,y^7/3$$

Soln for $T_w(z_b, y_b)$ - 1 - L18($\frac{6}{21}$)

Here, $\theta_w(z_b, y_b)$ is specified. Then

$$\theta_w = \sum_{i=1}^{N} (a_i \times g_i + c_{u_i} \times G_i)_{z_b,y_b} - (\frac{z_b^4 + y_b^4}{48})$$

In this case, let $a_i \equiv c_{tw,i}$. Then

$$\begin{aligned} \sum_{i=1}^{N} c_{tw,i} \times g_i \,|_{z_b,y_b} &= \theta_w - \sum_{i=1}^{N} c_{u_i} \times G_i \,|_{z_b,y_b} + (\frac{z_b^4 + y_b^4}{48}) \\ &= \Phi_{z_b,y_b} \text{ (say)} = \text{ known function} \end{aligned}$$

Therefore $c_{tw,i}$ can be determined by LU-decomposition.

Soln for $T_w\,(z_b, y_b)$ - 2 - L18($\frac{7}{21}$)

To determine $Nu_{tw}\,(z_b, y_b)$, we need to determine $q_w\,(z_b, y_b)$.

$$
\begin{aligned}
q_w\,(z_b, y_b) &= k\,\frac{\partial T}{\partial n}\,|_{z_b,y_b} = k\,(l\,\frac{\partial T}{\partial z} + m\,\frac{\partial T}{\partial y})\,|_{z_b,y_b} \\
(\frac{D_h^3}{8\ f\ Re})\,\frac{q_w}{\bar{q}_w} &= \frac{\partial \theta}{\partial n}\,|_{z_b,y_b} = (l\,\frac{\partial \theta}{\partial z} + m\,\frac{\partial \theta}{\partial y})\,|_{z_b,y_b} \\
&= \sum_{i=1}^{N} l\,(c_{tw,i}\,\frac{\partial g_i}{\partial z} + c_{u_i}\,\frac{\partial G_i}{\partial z})_{z_b,y_b} \\
&+ \sum_{i=1}^{N} m\,(c_{tw,i}\,\frac{\partial g_i}{\partial y} + c_{u_i}\,\frac{\partial G_i}{\partial y})_{z_b,y_b} \\
&- (\frac{l\,z_b^3 + m\,y_b^3}{12})
\end{aligned}
$$

where l and m are *direction cosines* at the boundary.

Soln for $T_w(z_b, y_b)$ - 3 - L18($\frac{8}{21}$)

Bulk temperature is determined by Num Integration as

$$\theta_b = \frac{\int_A u\,\theta\,dz\,dy}{\int_A u\,dz\,dy}$$

Then,

$$\begin{aligned} Nu_{tw}(z_b, y_b) &= \left(\frac{q_w}{T_w - T_b}\right) \times \frac{D_h}{k} = \left(\frac{D_h^3}{8\,f\,Re}\right)\left(\frac{q_w}{\overline{q}_w}\right) \times \left(\frac{D_h}{\theta_w - \theta_b}\right) \\ &= \frac{\partial \theta}{\partial n}\Big|_{z_b, y_b} \times \left(\frac{D_h}{\theta_w - \theta_b}\right) \\ \overline{Nu}_{tw} &= \frac{1}{S}\oint Nu_{tw}\,ds \end{aligned}$$

where S is the duct perimeter.

Soln for $q_w\,(z_b, y_b)$ - 1 - L18($\frac{9}{21}$)

Here, let $a_i \equiv c_{qw,i}$. Then, from slide 7,

$$\begin{aligned}
\sum_{i=1}^{N} c_{qw,i}\,(l\,\frac{\partial g_i}{\partial z} + m\,\frac{\partial g_i}{\partial y})_{z_b,y_b} &= (\frac{D_h^3}{8\,f\,Re})\,(\frac{q_w}{\bar{q}_w}) + (\frac{l\,z_b^3 + m\,y_b^3}{12}) \\
&- \sum_{i=1}^{N} c_{u_i}\,(l\,\frac{\partial G_i}{\partial z} + m\,\frac{\partial G_i}{\partial y})_{z_b,y_b} \\
&= \Omega_{z_b,y_b}\ \text{(say)} = \text{ known function}
\end{aligned}$$

Now, define

$$f_i \equiv (l\,\frac{\partial g_i}{\partial z} + m\,\frac{\partial g_i}{\partial y})_{z_b,y_b}\quad \text{Known}$$

Hence, $\sum_{i=1}^{N}\ c_{qw,i} \times f_i\,|_{z_b,y_b} = \Omega_{z_b,y_b}$. Therefore $c_{qw,i}$ can be determined by LU-decomposition.

Soln for $q_w(z_b, y_b)$ - 2 - L18($\frac{10}{21}$)

Therefore, the solutions is

$$\theta = \sum_{i=1}^{N} (c_{qw,i} \times g_i + c_{u_i} \times G_i)_{z,y} - (\frac{z^4 + y^4}{48})$$

$$\theta_w = \sum_{i=1}^{N} (c_{qw,i} \times g_i + c_{u_i} \times G_i)_{z_b,y_b} - (\frac{z_b^4 + y_b^4}{48})$$

$$\overline{\theta}_w = \frac{1}{S} \oint \theta_w \, ds \quad \rightarrow \overline{q}_w = \frac{1}{S} \oint q_w \, ds$$

Now, after evaluating θ_b,

$$Nu_{qw}(z_b, y_b) = (\frac{D_h^3}{8 \, f \, Re})(\frac{q_w}{\overline{q}_w}) \times (\frac{D_h}{\theta_w - \theta_b})$$

$$\overline{Nu}_{qw} = (\frac{D_h^3}{8 \, f \, Re}) \times (\frac{D_h}{\overline{\theta}_w - \theta_b})$$

Soln for $h_w\,(z_b, y_b)$ - 1 - L18($\frac{11}{21}$)

In this case, $q_w\,(z_b, y_b) = k\,(\partial T/\partial n)_{z_b,y_b} = h_w\,(T_w - T_\infty)$.
Therefore, with $a_i \equiv c_{hw,i}$, we have

$$\theta = \sum_{i=1}^{N} (c_{hw,i} \times g_i + c_{u_i} \times G_i)_{z,y} - (\frac{z^4 + y^4}{48})$$

$$\theta_w = \sum_{i=1}^{N} (c_{hw,i} \times g_i + c_{u_i} \times G_i)_{z_b,y_b} - (\frac{z_b^4 + y_b^4}{48})$$

$$(\frac{\partial \theta}{\partial n})_{z_b,y_b} = \frac{h_w}{k}\,(\theta_w - \theta_\infty)$$

$$= \sum_{i=1}^{N} c_{hw,i}\,(l\,\frac{\partial g_i}{\partial z} + m\,\frac{\partial g_i}{\partial y})_{z_b,y_b}$$

$$+ \sum_{i=1}^{N} c_{u_i}\,(l\,\frac{\partial G_i}{\partial z} + m\,\frac{\partial G_i}{\partial y})_{z_b,y_b} - (\frac{l\,z_b^3 + m\,y_b^3}{12})$$

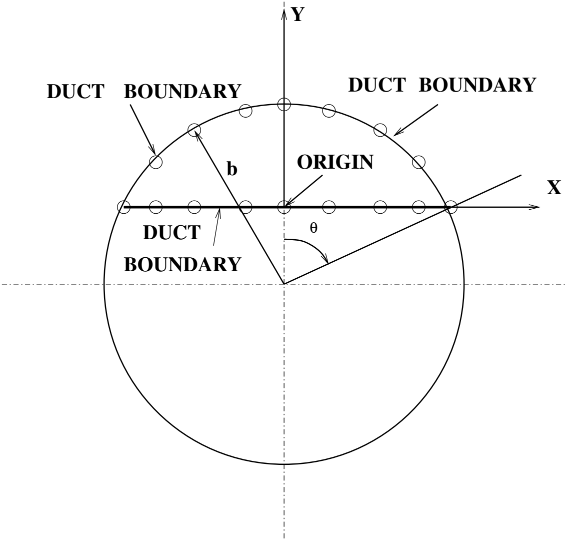
Soln for $h_w\,(z_b, y_b)$ - 2 - L18($\frac{12}{21}$)

Substituting for θ_w in Eqn for $(\partial\theta/\partial n)_{z_b,y_b}$, it can be shown that

$$
\begin{aligned}
\sum_{i=1}^{N} c_{hw,i}\, F_i &= \left(\frac{l\, z_b^3 + m\, y_b^3}{12}\right) - \frac{h_w}{k}\left(\frac{z_b^4 + y_b^4}{48} - \theta_\infty\right) \\
&- \sum_{i=1}^{N} c_{u_i}\left(l\,\frac{\partial G_i}{\partial z} + m\,\frac{\partial G_i}{\partial y} - \frac{h_w}{k}\,G_i\right)_{z_b,y_b} \\
&= \Gamma_{z_b,y_b}\ \text{(say)} = \text{ known function} \\
\text{where }\ F_i &= \left(l\,\frac{\partial g_i}{\partial z} + m\,\frac{\partial g_i}{\partial y} - \frac{h_w}{k}\,g_i\right)_{z_b,y_b}
\end{aligned}
$$

Now, $c_{hw,i}$ are determined by LU decomposition from $\sum_{i=1}^{N}\ c_{hw,i}\,F_i = \Gamma_{z_b,y_b}$.
Hence, θ_w and $q_w = (\partial\theta/\partial n)_{z_b,y_b} \times (8\, fRe\, \overline{q}_w/D_h^3)$ are determined.

Circular Segment Cross-Section - L18($\frac{13}{21}$)



17 points are chosen. b = radius, θ = Apex angle

Local Nu - T_w = const (θ = 90) - L18($\frac{14}{21}$)

i	z_b	y_b	l	m	q_w	Nu_{tw}
1	-1.0	0.0	0.0	-1.0	0.804E-04	0.023
2	-0.99	0.141	-0.99	0.141	0.380E-02	1.09
3	-0.75	0.661	-0.75	0.661	0.148E-01	4.24
4	-0.5	0.866	-0.5	0.866	0.181E-01	5.17
5	-0.25	0.968	-0.25	0.968	0.194E-01	5.55
14	0.0	0.0	0.0	-1.0	0.247E-01	7.06
15	-0.35	0.0	0.0	-1.0	0.207E-01	5.93
16	-0.75	0.0	0.0	-1.0	0.846E-02	2.42
17	-0.99	0.0	0.0	-1.0	0.359E-03	0.103

Exploiting symetry about z = 0, values for negative z are shown.
Low vaues of q_w correspond to hot-spot regions. $\overline{Nu}_{tw}$ = 4.02.

Effect of θ - T_w = const - L18($\frac{15}{21}$)

	θ degrees				
c_i	90	60	45	30	10
c_3	-0.247e-1	-0.400e-2	0.893e-3	-0.928e-4	-0.136e-6
c_4	0.238e-7	0.239e-6	-0.450e-7	-0.345e-7	0.319e-8
c_7	-0.243e-1	-0.159e-1	-0.103e-1	-0.514e-2	-0.627e-3
c_8	0.833e-6	-0.613e-6	0.125e-5	0.563e-6	0.479e-7
c_{11}	0.305e-2	0.275e-2	0.295e-2	0.340e-2	0.405e-2
c_{12}	-0.245e-5	-0.154e-6	-0.599e-5	-0.314e-5	-0.334e-4
c_{15}	0.358e-3	0.35-e-3	0.587e-3	0.931e-3	0.147e-2
c_{16}	0.175-5	0.733e-6	0.744e-5	0.562e-5	0.624e-3
c_{17}	-0.105e-6	-0.111e-5	-0.490e-5	-0.141e-4	0.449e-3
$\overline{Nu}_{tw}$	4.02	3.90	3.79	3.68	3.04

$\theta = 90$ corresponds to a duct of semi-circular cross section.

Local Nu - q_w = const (θ = 60) - L18($\frac{16}{21}$)

i	z_b	y_b	l	m	T_w	Nu_{qw}
1	-0.866	0.0	0.0	-1.0	0.237e-2	0.610
2	-0.857	0.0147	-0.857	0.141	0.231e-2	0.628
3	-0.65	0.26	-0.65	0.661	0.125e-2	1.22
4	-0.433	0.401	-0.433	0.866	0.544e-3	3.25
5	-0.217	0.476	-0.217	0.968	0.159E-3	37.5
14	0.0	0.0	0.0	-1.0	0.0	-11.2
15	-0.303	0.0	0.0	-1.0	0.405e-3	4.84
16	-0.65	0.0	0.0	-1.0	0.157e-2	0.944
17	-0.857	0.0	0.0	-1.0	0.234E-2	0.619

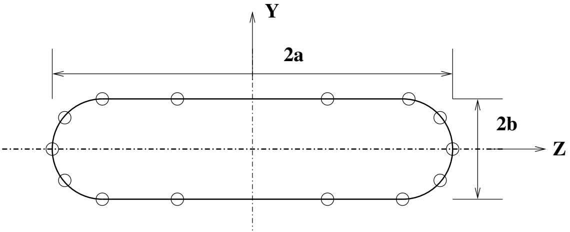
Exploiting symetry about z = 0, values for negative z are shown. In this case, T_b = 0.000122. Hence, at $z_b = y_b = 0$, Nu is negative. $\overline{Nu}_{qw}$ = 1.657

Effect of θ - q_w = const - L18($\frac{17}{21}$)

	θ degrees				
c_i	90	60	45	30	10
c_3	-0.143e-1	-0.212e-2	-0.449e-3	-0.444e-4	-0.667e-7
c_4	0.173e-1	0.470e-2	0.164e-2	0.394e-3	0.520e-5
c_7	-0.353e-1	-0.193e-1	-0.116e-1	-0.546e-2	-0.631e-3
c_8	-0.490e-2	-0.200e-2	-0.827e-3	-0.632e-3	-0.720e-4
c_{11}	0.403e-2	0.385e-2	0.391e-2	0.402e-2	0.415e-2
c_{12}	-0.147e-3	-0.859e-3	-0.190e-2	0.433e-3	0.473e-3
c_{15}	0.665e-3	0.502e-3	0.479e-3	0.521e-3	0.593e-3
c_{16}	0.178-3	0.226e-2	0.285e-2	0.110e-3	0.397e-3
c_{17}	0.781e-4	0.139e-3	0.124e-3	0.917e-4	0.807e-5
$\overline{Nu}_{qw}$	2.78	1.657	1.03	0.433	0.0495

$\theta = 90$ corresponds to a duct of semi-circular cross section.
Note that $\overline{Nu}_{qw} < \overline{Nu}_{tw}$ for all angles.

Rectangular Duct - Rounded Side - L18($\frac{18}{21}$)



14 points are chosen. b = radius, 2a = Long side
b = a corrsponds to the circular duct.

Effect of b/a - L18($\frac{19}{21}$)

b/a	c_{u1}	c_{u4}	c_{u8}	c_{u12}	fRe
0.25	0.0292	0.262	-0.0372	-0.00358	19.78
0.50	0.110	0.182	-0.0597	0.0179	17.23
1.0	0.25	0.0	0.0	0.0	16.0
b/a	c_{t1}	c_{t4}	c_{t8}	c_{t12}	$\overline{Nu}_{tw}$
0.25	-0.693e-3	-0.783e-2	0.111e-2	0.113e-2	5.944
0.50	-0.940e-2	-0.176e-1	0.849e-2	-0.943e-3	4.73
1.0	-0.469e-1	0.0	0.521e-2	0.0	4.367
b/a	c_{q2}	c_{q4}	c_{q8}	c_{q12}	$\overline{Nu}_{qw}$
0.25	-0.348e-3	-0.432e-2	-0.874e-3	0.158e-2	-15.46
0.50	-0.253e-2	-0.929-2	0.276e-2	0.125e-2	5.056
1.0	0.0	0.0	0.521e-2	0.0	4.367

As b/a $\rightarrow$ 0, fRe $\rightarrow$ 24. For b/a = 1.0, $\overline{Nu}_{qw} = \overline{Nu}_{tw}$.
Negative $\overline{Nu}_{qw}$ at b/a = 0.25 indicates $\overline{T}_w < T_b$.

Special Case b/a =1 - L18($\frac{20}{21}$)

Here let $q_w = \overline{q}_w \, (1 + A\,cos(\theta))$ $\overline{q}_w$ = 0.0625.

	A = 0.2			A = 0.5		
θ	q_w	$Nu_{\theta,exact}$	Nu_θ	q_w	$Nu_{\theta,exact}$	Nu_θ
0	0.075	3.65	3.65	0.0938	3.13	3.13
60	0.0688	3.94	3.94	0.0781	3.53	3.53
90	0.0625	4.365	4.365	0.0625	4.365	4.365
120	0.0563	5.03	5.03	0.0469	7.20	7.20
180	0.05	6.20	6.20	0.0313	-23.9	-24.0
-60	0.0688	3.94	3.94	0.0781	3.53	3.53
-90	0.0625	4.365	4.365	0.0625	4.365	4.365
-120	0.0563	5.03	5.03	0.0469	7.20	7.20

For A = 0.5, Negative Nu indicates $T_{w,\theta} < T_b$

Conclusions - L18($\frac{21}{21}$)

1. Solutions for circumferntial variation of h_w are not given here. This is left as in exercise that will require writing a general computer program with LU decomposition
2. The method described allows for maximum 17 boundary points. But more points can be taken for greater accuracy in some ducts of complex cross-section. Functions g_i and G_i for i $>$ 17 must be generated.
3. In the next lecture, we shall consider Developing Heat Transfer situations.