ME-662 CONVECTIVE HEAT AND MASS TRANSFER

A. W. Date
Mechanical Engineering Department
Indian Institute of Technology, Bombay
Mumbai - 400076
India

LECTURE-20 SUPERPOSITION TECHNIQUE

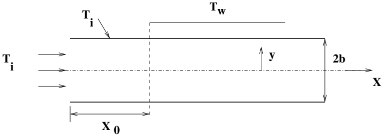
LECTURE-20 SUPERPOSITION TECHNIQUE

1. Effect of Axially varying thermal boundary condition on developing heat transfer Pr $>>$ 1
2. Axial variation of T_w
3. Axial variation of q_w

Axial Variation of T_w - L20($\frac{1}{12}$)

Our thermal entry length solution for Pr $>>$1 may be viewed as solution to a step-function ($T_w - T_i$) at x = x_0 . Thus, for $x \geq x_0$

$$T - T_i = [1 - \theta\,(x^* - x_0^*, y^*)]\;(T_w - T_i) \quad \rightarrow \theta = \frac{T - T_w}{T_i - T_w}$$



Therefore, for arbitrary variation of T_w , we have

$$T - T_i \;=\; \int_o^{x^*} [1 - \theta\,(x^* - x_0^*, y^*)]\,\frac{dT_w}{dx_0^*}\,dx_0^* \;+\; \sum_{k=1}^{NK} \left[1 - \theta\,(x^* - x_{0,k}^*, y^*)\right]\,\Delta(T_w - T_i)_k$$

Further Development - 1 - L20($\frac{2}{12}$)

Therefore, the wall heat flux is evaluated as

$$\begin{aligned} q_w\left(x^*\right) &= k\,\frac{\partial T}{\partial y}\,|_{y=b} = \frac{k}{b}\,\frac{\partial T}{\partial y^*}\,|_{y^*=1} \\ &= -\,\frac{k}{b}\left[\int_o^{x^*}\,\theta^{'}\left(x^* - x_0^*, 1\right)\frac{dT_w}{dx_0^*}\,dx_0^*\right. \\ &+ \left.\sum_{k=1}^{NK}\theta^{'}\left(x^* - x_{0,k}^*, 1\right)\Delta(T_w - T_i)_k\right] \end{aligned}$$

But, we know that (see lecture 19)

$$\theta^{'}\left(1\right) = -\sum_{n=0}^{\infty} A_n\, exp(-\,\frac{8}{3}\,\lambda_n^2\,x^*) \rightarrow A_n = -\,C_n\,Y_n^{'}\left(1\right)$$

Further Development - 2 - L20($\frac{3}{12}$)

Therefore, substitution gives

$$
\begin{aligned}
q_w\left(x^*\right) &= \frac{k}{b}\left[\int_o^{x^*} \sum_{n=0}^{\infty} A_n\, exp\left\{-\frac{8}{3}\,\lambda_n^2\left(x^* - x_0^*\right)\right\} \frac{dT_w}{dx_0^*}\, dx_0^* \right. \\
&\left. + \sum_{k=1}^{NK} \sum_{n=0}^{\infty} A_n\, exp\left\{-\frac{8}{3}\,\lambda_n^2\left(x^* - x_{0,k}^*\right)\right\} \Delta(T_w - T_i)_k \right] \\
T_b - T_i &= \frac{4b}{k}\int_0^{x^*} q_w\left(x^*\right) dx^* \\
Nu_{x^*} &= \frac{h_x\left(4b\right)}{k} = \frac{q_w\left(x^*\right)}{\left(T_w - T_b\right)_{x^*}} \times \frac{4b}{k}
\end{aligned}
$$

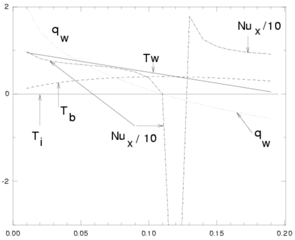
A Problem - L20($\frac{4}{12}$)

Let $T_w - T_i = (A + B\,x^*)$ $\rightarrow dT_w/dx_0^* = B$. Then

$$
\begin{aligned}
q_w\,(x^*) &= \frac{k}{b}\left[\frac{3B}{8}\sum_{n=0}^{\infty}\frac{A_n}{\lambda_n^2}\left\{1 - exp(-\frac{8}{3}\,\lambda_n^2\,x^*)\right\}\right. \\
&\quad \left. + \; A\sum_{n=0}^{\infty} A_n\,exp(-\frac{8}{3}\,\lambda_n^2\,x^*)\right] \quad (\text{note } x_0 = 0) \\
T_w - T_b &= \frac{9B}{16}\sum_{n=0}^{\infty}\frac{A_n}{\lambda_n^4}\left\{1 - exp(-\frac{8}{3}\,\lambda_n^2\,x^*)\right\} \\
&\quad + \; \frac{3A}{2}\sum_{n=0}^{\infty} A_n\,exp(-\frac{8}{3}\,\lambda_n^2\,x^*) \\
Nu_{x^*} &= \frac{q_w\,4b}{k(T_w - T_b)} \quad \text{as x } \rightarrow \infty,\, Nu_x = \frac{8\,\sum_0^{\infty}\;A_n/\lambda_n^2}{3\,\sum_0^{\infty}\;A_n/\lambda_n^4} = 8.235
\end{aligned}
$$

Results for A = 1 and B = -5 - L20($\frac{5}{12}$)

x^*	T_w	T_b	q_w	Nu_x
0	1.0	0	8.7	35
.05	.75	.32	.73	6.8
.10	.50	.39	.1	3.7
.11	.45	.4	0	.04
.12	.40	.4	-.1	-66
.15	.25	.37	-0.32	12.5
.17	.15	.34	-0.45	9.7
.19	.05	.30	-0.56	9.2
.20	0	.27	-.62	9.1



Strange things happen. T_w reduces to $T_i = 0$ at $x^* = 0.2$. T_b increases from 0 till $x^* = 0.11$ but then falls. $q_w > 0$ for $x^* \leq 0.11$ but then turns negative resulting in negative Nu which then again rises to Nu = 18 at $x^* = 0.13$ and then again falls. For $x^* > 0.12$, $T_b > T_w$.

Axial Variation of q_w - L20($\frac{6}{12}$)

From lecture 19, we know that the temperature response for **step-jump** in q_w at $x^* = x_0^*$ is given by

$$\begin{aligned}
\Psi &= \frac{T - T_i}{q_w\, b/k} \\
&= \frac{3}{4}\,({y^*}^2 - \frac{{y^*}^4}{6}) + 4\,x^* - \frac{39}{280} \\
&+ \sum_{n=1}^{\infty} C_n\, Y_n\,(y^*)\,\exp\,(-\frac{8}{3}\,\lambda_n^2\,x^*) \\
\Psi_w &= \frac{17}{35} + 4\,x^* + \sum_{n=1}^{\infty} B_n\,\exp\,(-\frac{8}{3}\,\lambda_n^2\,x^*) \\
\frac{\partial \Psi}{\partial x^*} &= 4 - \frac{8}{3}\sum_{n=1}^{\infty} B_n\,\lambda_n^2\,\exp\,(-\frac{8}{3}\,\lambda_n^2\,x^*)
\end{aligned}$$

where $B_n = C_n\, Y_n\,(1)$

Further Development - 1 - L20($\frac{7}{12}$)

Here, we consider only continuous variation of $q_w\,(x^*)$. Then, response of bulk and wall temperature will be

$$
\begin{aligned}
T_w - T_i &= \frac{b}{k}\int_0^{x^*} \frac{\partial \Psi}{\partial x_0^*}\, q_w\,(x_0^*)\, dx_0^* \\
&= \frac{b}{k}\int_0^{x^*} \left[4 - \frac{8}{3}\sum_{n=1}^{\infty} B_n\,\lambda_n^2 \exp\left(-\frac{8}{3}\,\lambda_n^2\, x^*\right)\right]\, q_w\,(x_0^*)\, dx_0^* \\
T_b - T_i &= \frac{4b}{k}\int_0^{x^*} q_w\,(x_0^*)\, dx_0^* \\
Nu_{x^*} &= \frac{q_w(x^*)}{T_w - T_b} \times \frac{4b}{k}
\end{aligned}
$$

A Problem - L20($\frac{8}{12}$)

In nuclear reactors, the fuel elements (rods or plates) generate sinusoidally varying heat flux along the cooling channels. Thus, let

$$\frac{q_w}{q_{w,max}} = \sin\left(\frac{\pi\, x}{L}\right)$$

where L is the length of the cooling channel. Then

$$\begin{aligned} \frac{T_b - T_i}{(q_{w,max}\, b/k)} &= \int_0^{x^*} 4 \sin\left(\frac{\pi\, x_0^*}{L^*}\right) dx_0^* \\ &= \left(\frac{4\, L^*}{\pi}\right) \left[1 - \cos\left(\frac{\pi\, x^*}{L^*}\right)\right] \end{aligned}$$

Problem Contd. - 1 - L20($\frac{9}{12}$)

$$
\begin{aligned}
\frac{T_w - T_i}{(q_{w,max}\, b/k)} &= \int_0^{x^*} 4 \sin\left(\frac{\pi\, x_0^*}{L^*}\right) dx_0^* \\
&- \frac{8}{3} \int_0^{x^*} \sum_{n=1}^{\infty} B_n\, \lambda_n^2 \exp\left(-\frac{8}{3}\, \lambda_n^2\, x^*\right) \sin\left(\frac{\pi\, x_0^*}{L^*}\right) dx_0^* \\
&= \left(\frac{4\, L^*}{\pi}\right) \left[1 - \cos\left(\frac{\pi\, x^*}{L^*}\right)\right] \\
&+ \sum_{n=1}^{\infty} \left[\frac{B_n}{1 + \{(3\,\pi)/(8\,\lambda_n^2\, L^*)\}^2}\right. \\
&\times \left[\sin\left(\frac{\pi\, x^*}{L^*}\right) \exp\left(-\frac{8}{3}\, \lambda_n^2\, x^*\right)\right. \\
&+ \left.\left.\frac{3\,\pi}{8\, L^*} \left\{\cos\left(\frac{\pi\, x^*}{L^*}\right) \exp\left(-\frac{8}{3}\, \lambda_n^2\, x^*\right) - 1\right\}\right]\right]
\end{aligned}
$$

Problem Contd. - 2 - L20($\frac{10}{12}$)

From the results of last two slides

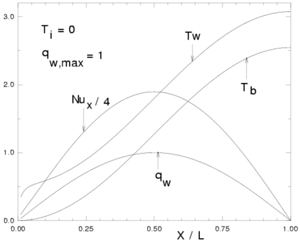
$$\begin{aligned}
\frac{T_w - T_b}{(q_{w,max}\, b/k)} &= \sum_{n=1}^{\infty} \left[\frac{B_n}{1 + \{(3\,\pi)/(8\,\lambda_n^2\, L^*)\}^2} \right. \\
&\times \left[\sin\left(\frac{\pi\, x^*}{L^*}\right) \exp\left(-\frac{8}{3}\,\lambda_n^2\, x^*\right) \right. \\
&+ \left. \left. \frac{3\,\pi}{8\,L^*} \left\{ \cos\left(\frac{\pi\, x^*}{L^*}\right) \exp\left(-\frac{8}{3}\,\lambda_n^2\, x^*\right) - 1 \right\} \right] \right] \\
Nu_x &= \frac{4 \sin(\pi\, x^*/L^*)}{(T_w - T_b)/(q_{w,max}\, b/k)}
\end{aligned}$$

Values of λ_n and B_n are given in lecture 19.

Results - L20($\frac{11}{12}$)

$q_w = q_{w,max} \sin\left(\pi\, x^*/L^*\right)$
$q_{w,max} = 1$

x/L	q_w	T_b	T_w	$\frac{Nu_x}{4}$
.01	.031	6e-3	.35	.09
.05	.156	.016	.525	.307
.10	.31	.062	.595	.580
.25	.707	.373	.908	1.32
.50	1.0	1.27	1.81	1.87
.70	.809	2.02	2.56	1.51
.80	.588	2.30	2.84	1.10
.90	.309	2.48	3.02	.577
.95	.156	2.53	3.06	.292
1.0	0.0	2.55	3.04	0.0



Note that $Nu_{x,max}$ = 7.48 occurs at x/L = 0.5 where $q_{w,max}$ occurs, but $T_{w,max}$ and $T_{b,max}$ occur at x/L = 0.95. This problem is of relevance to Nuclear reactors.

Summary - L20($\frac{12}{12}$)

1. We have considered fully developed heat transfer in circular tube and annuli and parallel plates.
2. We have also presented a general method for flow and heat transfer in *singly connected* ducts of arbitrary cross-section and arbitrary variations of T_w, q_w and h_w
3. We presented developing heat transfer solutions for circular tube and parallel plates for $q_w\,(x)$ = const and $T_w\,(x)$ = const for the entire range of Prandtl numbers.
4. Finally, we extrapolated these solutions to situations involving arbitrary axial variations of heat flux and wall temperature. However, for complex ducts, it is best to adopt CFD solutions
5. This completes discussion on Laminar duct flow heat transfer.