ME-662 CONVECTIVE HEAT AND MASS TRANSFER

A. W. Date
Mechanical Engineering Department
Indian Institute of Technology, Bombay
Mumbai - 400076
India

LECTURE-5 LAMINAR BOUNDARY LAYERS

LECTURE-5 LAMINAR BLs

1. 2D Flow and Scalar Transport Equations
2. Boundary Layer Approximations
3. 2D Velocity Boundary Layer Equations
4. 2D Temperature and Concentration Boundary Layer Equations
5. Methods of Solutions

3D Navier Stokes Equations - L5($\frac{1}{15}$)

Mass Conservation equation

$$\frac{\partial(\rho_m)}{\partial t} + \frac{\partial(\rho_m \, u_j)}{\partial x_j} = 0 \tag{1}$$

Momentum equation in X_i direction (3 equations)

$$\begin{aligned} \frac{\partial(\rho_m \, u_i)}{\partial t} + \frac{\partial(\rho_m \, u_j \, u_i)}{\partial x_j} &= -\frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\mu \frac{\partial u_i}{\partial x_j}\right] \\ &+ \rho_m \, B_i + \frac{\partial}{\partial x_j}\left[\mu \frac{\partial u_j}{\partial x_i}\right] \end{aligned} \tag{2}$$

2D Flow Assumptions - L5($\frac{2}{15}$)

Consider a 2D forms of Navier-Stokes Equations with following assumptions

1. Flow is Steady ($\partial/\partial t = 0$)
2. Flow is Laminar
3. Fluid properties ρ, Cp, μ, k and D are uniform
4. Independent variables are x = x_1 and y = x_2
5. Dependent variables are u = u_1 and v = u_2
6. Body Forces are neglected

2D Flow Equations L5($\frac{3}{15}$)

Continuity Equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

x-Momentum Equation

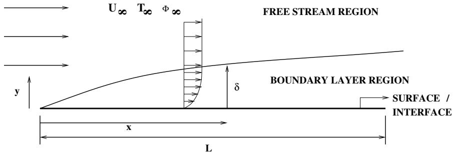
$$\rho \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] \tag{4}$$

y-Momentum Equation

$$\rho \left[u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right] = -\frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right] \tag{5}$$

BL Concept L5($\frac{4}{15}$)

1. The concept of the wall boundary layer was first introduced by L. Prandtl in 1904 to theoretically predict the *drag* experienced by a body immersed in a flowing fluid.
2. Prandtl identified a thin viscosity affected region close to a surface in which significant velocity variations take place.
3. Outside the Boundary Layer, Free Stream is Inviscid



Define

1. $x^* = x/L,\ y^* = y/L$
2. $u^* = u/U_\infty,\ v^* = v/U_\infty$
3. $p^* = p/(\rho U_\infty^2)$, Re = $\frac{u_\infty L}{\nu}$
4. $\delta << X, u >> v$

L and U_∞ are reference length & velocity

Non-Dimensionalised Equations L5($\frac{5}{15}$)

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \qquad (6)$$

$$\frac{1}{1} + \frac{\delta^*}{\delta^*}$$

$$u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re}\left[\frac{\partial^2 u^*}{\partial {x^*}^2} + \frac{\partial^2 u^*}{\partial {y^*}^2}\right] \qquad (7)$$

$$1\frac{1}{1} + \delta^*\frac{1}{\delta^*} = O\left(1\right) + ({\delta^*}^2)\left[\frac{1}{1^2} + \frac{1}{{\delta^*}^2}\right]$$

$$u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re}\left[\frac{\partial^2 v^*}{\partial {x^*}^2} + \frac{\partial^2 v^*}{\partial {y^*}^2}\right] \qquad (8)$$

$$1\frac{\delta^*}{1} + \delta^*\frac{\delta^*}{\delta^*} = \delta^* + ({\delta^*}^2)\left[\frac{\delta^*}{1^2} + \frac{\delta^*}{{\delta^*}^2}\right]$$

BL Approximations L5($\frac{6}{15}$)

$$
\begin{aligned}
u^* &>> v^* \\
\frac{\partial u^*}{\partial y^*} &>> \frac{\partial u^*}{\partial x^*}, \frac{\partial v^*}{\partial x^*}, \frac{\partial v^*}{\partial y^*} \\
\frac{\partial^2 u^*}{\partial {y^*}^2} &>> \frac{\partial^2 u^*}{\partial {x^*}^2} \\
\frac{\partial^2 v^*}{\partial {y^*}^2} &>> \frac{\partial^2 v^*}{\partial {x^*}^2} \\
\frac{\partial p^*}{\partial y^*} &\simeq O\left(\delta^*\right) \text{ negligible} \\
\frac{\partial p^*}{\partial x^*} &\simeq O\left(1\right) = \frac{dp^*_\infty}{dx^*} = \frac{dp^*_w}{dx^*}
\end{aligned}
$$

2D BL Equations L4($\frac{7}{15}$)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

$$\rho \left[u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right] = -\frac{dp_\infty}{d\,x} + \mu \frac{\partial^2 u}{\partial y^2} \tag{10}$$

$$-\frac{dp_\infty}{d\,x} = \rho U_\infty \frac{d\,U_\infty}{d\,x} \qquad U_\infty(x) \text{ specified} \tag{11}$$

$$\text{local shear stress: } \tau_x = \mu \left\{ \frac{\partial u}{\partial y} \right\}_{y=0} \tag{12}$$

$$\text{average shear stress: } \overline{\tau} = \frac{1}{L} \int_0^L \mu \left\{ \frac{\partial u}{\partial y} \right\}_{y=0} d\,x \tag{13}$$

3D Energy Eqn L5($\frac{8}{15}$)

$$\rho_m \frac{D\,h_m}{D\,t} = \frac{\partial}{\partial x_j}\left[k_m \frac{\partial T}{\partial x_j}\right] - \frac{\partial(\sum m''_{j,k}\,h_k)}{\partial x_j} + \mu\,\Phi_v + \frac{D\,p}{D\,t} + \dot{Q}_{chem} + \dot{Q}_{rad} \quad (14)$$

where $h_m = \sum\ \omega_k\ h_k$ and $h_k = h^0_{f,k}\ (T_{ref}) + \int_{T_{ref}}^{T}\ Cp_k\ dT$

We again invoke uniform property assumption

Dimensionless 2D Energy Eqn L5($\frac{9}{15}$)

$$
\begin{aligned}
(u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*}) & \\
= & \ (\frac{1}{Re\,Pr}) \left[\frac{\partial^2 T^*}{\partial {x^*}^2} + \frac{\partial^2 T^*}{\partial {y^*}^2} \right] \\
+ & \ (Ec) \left\{ u^* \frac{\partial p^*}{\partial x^*} + v^* \frac{\partial p^*}{\partial y^*} \right\} + \dot{Q}^*_{chem} + \dot{Q}^*_{rad} \\
+ & \ (\frac{Ec}{Re}) \left[2\,(\frac{\partial u^*}{\partial x^*})^2 + 2\,(\frac{\partial v^*}{\partial y^*})^2 + (\frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*})^2 \right] \qquad (15)
\end{aligned}
$$

1. $T^* = (T - T_\infty)/\Delta T_o \ \rightarrow \ O\,(\,1\,)$
2. Pr = Prandtl Number = μ Cp / k = ν/α
3. Ec = Eckert Number = $U_\infty^2/Cp\Delta T_o$
4. $\dot{Q}^* = \dot{Q}\, L/(\rho_m\, Cp_m\, U_\infty)$

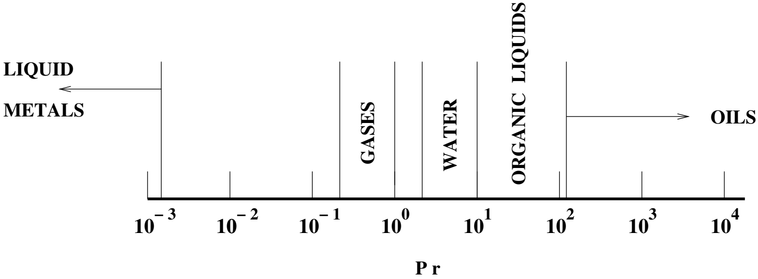
Energy Equation - B L Form L5($\frac{10}{15}$)

Carrying out Order-of-Magnitude analysis, and invoking B L approximations, we have

$$\rho C_p \left[u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] = k \frac{\partial^2 T}{\partial y^2} + \mu \left(\frac{\partial u}{\partial y} \right)^2 + u \frac{dp_\infty}{d\,x} + \dot{Q}_{chem} + \dot{Q}_{rad} \tag{16}$$

1. Note that $\partial^2 T^* / \partial {y^*}^2 >> \partial^2 T^* / \partial {x^*}^2$
2. In the viscous dissipation term, only $(\partial u^* / \partial y^*)^2$ is important.
3. In the pressure work terms, : $u\, dp_\infty / d\,x$ is important

Prandtl No Spectrum L5($\frac{11}{15}$)



Prandtl Number defines the Fluid Type

Mass Transfer Eqn L5($\frac{12}{15}$)

3D Mass Transfer Equation

$$\frac{\partial(\rho_m\,\omega_k)}{\partial t} + \frac{\partial(\rho_m\,u_j\,\omega_k)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\rho_m\,D\,\frac{\partial \omega_k}{\partial x_j}\right) + R_k \tag{17}$$

Carrying out Order-of-Magnitude analysis, invoking B L approximations, and making uniform property assumptions, we have

2D Mass Transfer B L Equation

$$\rho_m\left[u\,\frac{\partial \omega_k}{\partial x} + v\,\frac{\partial \omega_k}{\partial y}\right] = \rho_m\,D\,\frac{\partial^2 \omega_k}{\partial y^2} + R_k \tag{18}$$

Dimensionless Form L5($\frac{13}{15}$)

$$\left[u^* \frac{\partial \omega_k^*}{\partial x^*} + v^* \frac{\partial \omega_k^*}{\partial y^*}\right] = (\frac{1}{Re\ Sc}) \frac{\partial^2 \omega_k^*}{\partial y^{*2}} + R_k^* \tag{19}$$

1. $\omega^* = (\omega - \omega_\infty)/\Delta\omega_o \ \rightarrow$ O (1)
2. Sc = Schmidt Number = ν / D
3. R_k^* = $R_k\ L/(\rho_m\ U_\infty\ \Delta\omega_o)$

Summary L5($\frac{14}{15}$)

$$\frac{\partial(\rho\, u\, \Phi)}{\partial x} + \frac{\partial(\rho\, v\, \Phi)}{\partial y} = \frac{\partial}{\partial y}\left[\Gamma_\Phi\, \frac{\partial \Phi}{\partial y}\right] + S_\Phi \qquad (20)$$

Φ	Γ_Φ	S_Φ
1	0	0
u	μ_m	$-\, dp_\infty/d\, x$
T	k_m/Cp_m	$(\dot{Q}_{chem} + \dot{Q}_{rad} + \mu_m\, (\partial u/\partial y)^2 +\, u\, dp_\infty/d\, x)/Cp_m$
ω_k	$\rho_m\, D$	R_k

Recall: $a\, \Phi_{xx} + 2\, b\, \Phi_{xy} + c\, \Phi_{yy} = S\, (\Phi_x, \Phi_y, \Phi, x, y)$.
When the discriminant $b^2 - a\, c = 0$, the equation is parabolic.
When $b^2 - a\, c < 0$, the equation is elliptic.
When $b^2 - a\, c > 0$, the equation is hyperbolic.

Methods of Solution L5($\frac{15}{15}$)

Boundary Layer Equations are PARABOLIC .

There are 3 Methods of Solution

1. Similarity Method (PDE to ODE)
2. Integral Method (PDE to ODE)
3. Finite-Difference or Finite Element Method (PDE to Set of Algebraic Equations)