ME-662 CONVECTIVE HEAT AND MASS TRANSFER

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LECTURE-7 SIMILARITY SOLUTION TO VELOCITY BL

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1. Similarity Equation and Boundary Conditions
2. Shooting Method
3. Solutions to Velocity Boundary Layer Equation

Similarity Eqn and BCs -L7($\frac{1}{14}$)

Our interest is to solve

$$f''' + (\frac{m+1}{2})\, f\, f'' + m\,(1 - f'^2) = 0 \quad (1)$$

$$f\,(0) = -B_f\,(\frac{2}{m+1}) \quad f'\,(0) = 0 \quad \text{and} \quad f'\,(\infty) = 1 \quad (2)$$

The solution gives the velocity profiles

$$f'\,(\eta) = \frac{u}{U_\infty} = F\,(m, B_f) \quad (3)$$

$$\frac{v}{U_\infty}\, Re_x^{0.5} = -(\frac{m+1}{2})\left\{f + (\frac{m-1}{m+1})\,\eta\, f'\right\} \quad (4)$$

Parameters of Interest - L7($\frac{2}{14}$)

The $f^{'}(\eta)$ solution gives the Coefficient of Friction $C_{f,x}$ as a function of Reynolds number $Re_x = U_\infty\, x/\nu$

$$\tau_{w,x} = \mu \left\{ \frac{\partial u}{\partial y} \right\}_{y=0} = \mu\, U_\infty \sqrt{\frac{U_\infty}{\nu\, x}}\, f^{''}(0) \qquad (5)$$

$$C_{f,x} = \frac{\tau_{w,x}}{\rho U_\infty^2/2} = 2\, f^{''}(0)\, Re_x^{-0.5} \qquad (6)$$

$$\overline{C_f} = \frac{1}{L}\int_0^L \tau_{w,x}\, d\, x = \left(\frac{2}{3m+1}\right) C_{f,x} \qquad (7)$$

Therefore, we must determine $f^{''}(0)$. Further parameters of interest will be listed in a later slide.

Shooting Method - L7($\frac{3}{14}$)

The 3rd order equation is split into three 1st order ODEs

$$\frac{d\,f}{d\,\eta} = f' \quad \text{with} \quad f(0) = B_f\,(\frac{2}{m+1}) \quad \text{(known)} \tag{8}$$

$$\frac{d\,f'}{d\,\eta} = f'' \quad \text{with} \quad f'(0) = 0 \quad \text{(known)} \tag{9}$$

$$\frac{d\,f''}{d\,\eta} = f''' = -\left[(\frac{m+1}{2})\,f\,f'' + m\,(1 - f'^2)\right]$$

$$\text{with} \quad f''(0) \quad (\text{ unknown }) \tag{10}$$

Each equation is solved by Runge-Kutta Method from $\eta = 0$ to $\eta = \eta_{max}$ (in liu of $\eta = \infty$). Typically, $3 < \eta_{max} < 10$ suffices depending on the value of B_f and m.

Iterative Algorithm - L6($\frac{4}{14}$)

1. Select values of m and B_f
2. Select η_{max} and step change $d\eta$
3. Guess $f^{''}(0)$
4. Solve **three equations simulteneously** by R-K method
5. Check if value of $f^{'}(\eta_{max}) = 1$ or NOT
6. If NOT, revise $f^{''}(0) = \Phi$ as

$$\Phi(k+1) = \Phi(k) + (1 - \psi(k)) \left[\frac{\Phi(k) - \Phi(k-1)}{\psi(k) - \psi(k-1)}\right]$$

   where k is iteration number and $\psi = f^{'}(\eta_{max})$.

7. Go to step 4
8. At Convergence,
   1. Print values of $f(\eta), f^{'}(\eta), f^{''}(\eta)$.
   2. Note value of $f^{''}(0)$

Typical Convergence History - L7($\frac{5}{14}$)

Solution is obtained for m = 0, $B_f = 0$, $\eta_{max} = 7$ and $d\eta = \eta_{max}/300$. Intial Guess, $f^{''}(0) = 0.02$.

k	$f^{''}(0)$	$f(\eta_{max})$	$f^{'}(\eta_{max})$	$f^{''}(\eta_{max})$
1	0.02	0.465E+00	0.123E+00	0.115E-01
2	0.07	0.147E+01	0.342E+00	0.111E-01
3	0.220	0.382E+01	0.761E+00	0.125E-02
4	0.306	0.496E+01	0.948E+00	0.321E-03
5	0.329	0.525E+01	0.997E+00	0.221E-03
6	0.33071	0.527E+01	0.100E+01	0.216E-03

Because of very poor guess, 6 iterations are required.
In this case, $C_{f,x} = 0.6614\ Re_x^{-0.5}$ and $\overline{C_f} = 1.28\ Re_L^{-0.5}$.
Series Solution: $C_{f,x} = 0.664\ Re_x^{-0.5}$ and $\overline{C_f} = 1.328\ Re_L^{-0.5}$.

Typical Profiles - L7($\frac{6}{14}$)

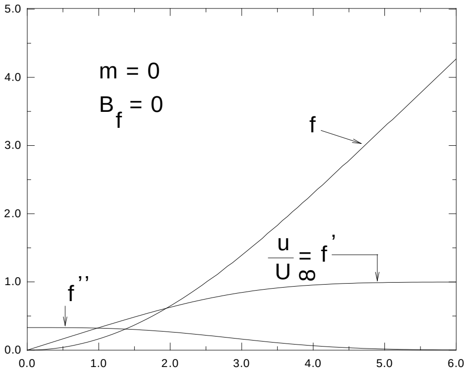


Figure: Profiles of f, f' and f'' - (m = 0 $B_f = 0$)

Characteristic Thicknesses - L7($\frac{7}{14}$)

1. The *Physical Thickness* δ is notionally asscociated with value of y where $u/U_\infty = f^{'}(\eta) \simeq 0.99$.
2. *Diplacement Thickness* δ_1 is defined as

$$\delta_1 = \int_0^\infty (1 - \frac{\rho u}{\rho_\infty U_\infty})\, d\, y \tag{11}$$

   It represents the Mass Deficit caused by the viscosity affected low velocity (that is $u < U_\infty$) region near a wall.
3. *Momentum Thickness* δ_2 is defined as

$$\delta_2 = \int_0^\infty \frac{u}{U_\infty} (1 - \frac{\rho u}{\rho_\infty U_\infty})\, d\, y \tag{12}$$

   It represents Momentum Deficit caused by the boundary layer

Dimensionless Forms - L7($\frac{8}{14}$)

In incompressible flows $\rho/\rho_\infty = 1$. Hence,

$$\delta^* = \frac{\delta}{x} Re_x^{0.5} \quad (13)$$

$$\delta_1^* = \frac{\delta_1}{x} Re_x^{0.5} = \int_0^\infty \left(1 - f'(\eta)\right) d\,\eta \quad (14)$$

$$\delta_2^* = \frac{\delta_2}{x} Re_x^{0.5} = \int_0^\infty f'(\eta)\left(1 - f'(\eta)\right) d\,\eta \quad (15)$$

$$C_{f,x} = \frac{\tau_{w,x}}{\rho U_\infty^2/2} = 2\,f''(0)\,Re_x^{-0.5} \quad (16)$$

These are evalutaed from Similarity solutions at convergence.

Effect of Pressure Gradient m - L7($\frac{9}{14}$)

Solutions with $B_f = 0$

m	β	$f''(0)$	δ^*	δ_1^*	δ_2^*	Remarks
4.000	1.600	2.396	1.330	0.340	0.157	
1.000	1.000	1.229	2.380	0.643	0.290	Stagnation
0.330	0.500	0.755	3.400	0.981	0.427	
0.000	0.000	0.330	4.900	1.727	0.663	Flat Plate
-0.040	-0.083	0.239	5.400	2.012	0.729	
-0.065	-0.139	0.163	5.800	2.330	0.786	
-0.085	-0.186	0.066	6.500	2.906	0.847	
-0.091	-0.200	0.000	7.420	3.498	0.868	Seperation

Excellent agreement with measurements of Nikuradze (1942) and Liepman and Dhawan (1951) for Flat Plate BL (m = 0)

Comments on Results - L7($\frac{10}{14}$)

1. For m = 0 (Flat Plate)
   $\delta^* \simeq 5$ and $f''(0) \simeq 0.33$
2. For $m > 0$ (Acc Flow)
   $\delta^* < 5$ and $f''(0) > 0.33$
3. For $m < 0$ (Dec Flow)
   $\delta^* > 5$ and $f''(0) < 0.33$
4. For $m \leq -0.091$ (Dec Flow) , $\delta^* > 5$ and $f''(0) \leq 0.$ Hence, Separation occurs

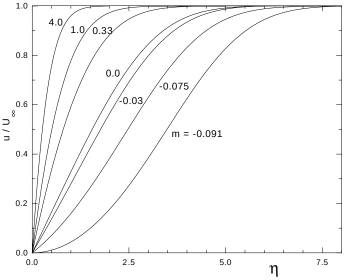


Figure: Velocity Profiles - Effect of m ($B_f = 0$)

Adv Pr Gr causes Flow thickening whereas Fav Pr Gr causes Flow thinning .

Effect of Suction/Blowing - L7($\frac{11}{14}$)

1. Recall that $B_f = (V_w(x)/U_\infty(x))\ Re_x^{0.5}$ = constant for similarity solutions to exist.
2. Therefore, since $U_\infty = C\,x^m$,

$$V_w \propto (\frac{U_\infty}{x})^{0.5} \propto x^{(m-1)/2} \quad (17)$$

3. Solutions obtained for m = 0 and m = 1 are shown on the next slide

Effect of B_f (m = 0) - L7($\frac{12}{14}$)

Flat Plate Flow

B_f	$f''(0)$	δ^*	δ_1^*	δ_2^*
-2.0	2.063	1.87	0.439	0.212
-1.0	1.155	2.80	0.728	0.336
-0.5	0.723	3.60	1.04	0.456
0.0	0.330	4.90	1.727	0.663
0.3	0.134	6.33	2.69	0.868
0.5	0.0351	8.40	4.406	1.07
0.612	0.0	-	-	-

1. $B_f < 0$ represents Suction
2. $B_f > 0$ represents Blowing
3. $B_f = 0.612$ represents Separation due to blowing

Effect of B_f (m = 1) - L7($\frac{13}{14}$)

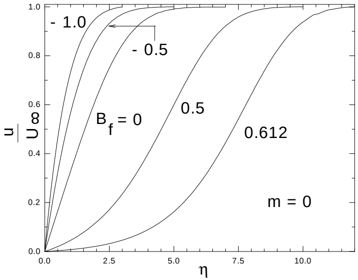
Stagnation Point Flow

B_f	$f''(0)$	δ^*	δ_1^*	δ_2^*
-2.0	2.611	1.4	0.337	0.161
-1.0	1.865	1.80	0.454	0.213
-0.5	1.53	2.07	0.538	0.247
0.0	1.229	2.38	0.643	0.290
0.3	1.069	2.59	0.719	0.320
0.5	0.972	2.73	0.776	0.342
1.0	0.763	3.16	0.939	0.403

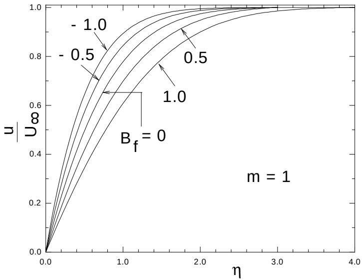
Even for $B_f = 1.0$, Separation does not occur

Velocity Profiles - L6($\frac{14}{14}$)

Flat Plate



Stagnation Point



Notice zero velocity gradient for $B_f = 0.612$