Classification of Partial Differential Equations

Q1.

A two-dimensional small-disturbance velocity potential equation for compressible flows is given as

$\left(1-M_\infty^2\right)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$, where M is the Mach number of flow.

(i) Examine whether this equation is parabolic, elliptic, or hyperbolic?

(ii) Justify your inference from pure physical arguments.

Solution

Consider the following second-order partial differential equation

$$A\phi_{xx}+B\phi_{xy}+C\phi_{yy}+D\phi_x+E\phi_y+F\phi+G=0$$

or, $$A\phi_{xx}+B\phi_{xy}+C\phi_{yy}+H=0 \qquad (1)$$

where $H=D\phi_x+E\phi_y+F\phi+G$

Assume that $\phi=\phi(x,y)$ is a solution of the differential equation.

It is to be noted that, the second-order derivatives along the characteristic curves corresponding to 2nd order partial differential equations are indeterminate and, indeed, they may be discontinuous across the characteristics. However, first derivatives are continuous functions of *x* and *y*. Thus,

$$d\phi_x=\frac{\partial\phi_x}{\partial x}dx+\frac{\partial\phi_x}{\partial y}dy=\phi_{xx}dx+\phi_{xy}dy \qquad (2)$$

$$d\phi_y=\frac{\partial\phi_y}{\partial x}dx+\frac{\partial\phi_y}{\partial y}dy=\phi_{yx}dx+\phi_{yy}dy \qquad (3)$$

From Eqs (1), (2) and (3), we have

$$\begin{bmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix}\begin{bmatrix} \phi_{xx} \\ \phi_{xy} \\ \phi_{yy} \end{bmatrix}=\begin{bmatrix} -H \\ d\phi_x \\ d\phi_y \end{bmatrix}$$

Since it is possible to have discontinuities in the second-order derivatives of the dependent variable across the characteristics, these derivatives are indeterminate. Therefore,

$$\begin{vmatrix} A & B & \mathrm{C} \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix}=0$$

or, $$A\left(\frac{dy}{dx}\right)^2-B\frac{dy}{dx}+C=0$$

Solving the equation yields the equations of the characteristics

$$\frac{dy}{dx}=\frac{B\pm\sqrt{B^2-4AC}}{2A}$$

Depending on the value of $B^2 - 4AC$, characteristic curves can be real or imaginary. For $B^2 - 4AC < 0$, no real characteristic is there and the equation is elliptic. For $B^2 - 4AC = 0$, one real characteristic is there and the equation is parabolic. When $B^2 - 4AC > 0$, two real characteristics are there and the equation is hyperbolic.

For the governing equation $\left(1 - M_\infty^2\right)\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$, we have

$A = 1 - M_\infty^2$, $B = 0$, and $C = 1$

Thus, $B^2 - 4AC = -4\left(1 - M_\infty^2\right)$.

If, $M_\infty < 1$, then $B^2 - 4AC < 0$ and the equation is elliptic. For $M_\infty = 1$, $B^2 - 4AC = 0$ and the equation is parabolic. For $M_\infty > 1$, then $B^2 - 4AC > 0$ and the equation is hyperbolic.

Q2.

Identify the nature of the following systems of partial differential equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = v$, where u and v are the two dependent variables.

Solution

$$\frac{\partial u}{\partial y} = v$$

Differentiating with respect to *y*, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial v}{\partial y}$$

Again, it is given that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$.

Comparing above two equations, we get

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial x}$$

or, $$u_{yy} - u_x = 0 \quad (4)$$

This becomes a second-order partial differential equation.

$$du_x = \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy = u_{xx}dx + u_{xy}dy \quad (5)$$

$$du_y = \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy = u_{xy}dx + u_{yy}dy \quad (6)$$

From Eqs (4), (5) and (6), we have

$$\begin{bmatrix} 0 & 0 & 1 \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix} \begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} u_x \\ du_x \\ du_y \end{bmatrix}$$

Since it is possible to have discontinuities in the second-order derivatives of the dependent variable across the characteristics, these derivatives are indeterminate. Therefore,

$$\begin{vmatrix} 0 & 0 & 1 \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0$$

or, $$(dx)^2 = 0$$

$$\therefore x = \text{constant}$$

Hence, there is one real characteristic and the given partial differential equation is parabolic in nature.

Q3.

Consider a general form of the energy conservation equation as:

$$\frac{\partial}{\partial t}\left(\rho C_p T\right) + \nabla.\left(\rho \vec{V} C_p T\right) = \nabla.\left(k \nabla T\right) + S \, .$$

In a physical problem, one is interested to obtain the transient temperature distribution (T as a function of x and t) in a uniform flow field ($u = U_\infty$=constant). Thermal diffusivity ($k/\rho C_p$)of the medium is negligibly small (can be taken as zero for the analysis). There is a uniform rate of volumetric heat generation (S) within the domain and in the physical space the temperature varies only along the x direction. The physical properties of the medium can be taken as invariants.

(i) Obtain an equation for the characteristics of the final simplified partial differential equation governing the above-mentioned physical problem.

(ii) Examine whether this equation is parabolic, elliptic or hyperbolic.

Solution

$$\frac{\partial}{\partial t}\left(\rho C_p T\right) + \nabla.\left(\rho \vec{V} C_p T\right) = \nabla.\left(k \nabla T\right) + S$$

$$\frac{\partial T}{\partial t} + U_\infty \frac{\partial T}{\partial x} = S$$

Differentiating with respect to x, we get

$$\frac{\partial^2 T}{\partial t \partial x} + U_\infty \frac{\partial^2 T}{\partial x^2} = 0 \qquad (7)$$

Differentiating with respect to t, we get

$$\frac{\partial^2 T}{\partial t^2} + U_\infty \frac{\partial^2 T}{\partial x \partial t} = 0 \qquad (8)$$

Multiplying Eq. (7) by U_∞ and then subtracting Eq. (8) [$U_\infty \times$Eq. (7)-Eq. (8)], we have

$$U_\infty^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 T}{\partial t^2}$$

This is the equation for the characteristics. This is a second-order partial differential equation.

The above equation can be written as

$$U_\infty^2 T_{xx} - T_{tt} = 0 \qquad (9)$$

$$dT_x = \frac{\partial T_x}{\partial x} dx + \frac{\partial T_x}{\partial t} dt = T_{xx} dx + T_{xt} dt \qquad (10)$$

$$dT_t = \frac{\partial T_t}{\partial x} dx + \frac{\partial T_t}{\partial t} dt = T_{tx} dx + T_{tt} dt \qquad (11)$$

From Eqs (9), (10) and (11), we have

$$\begin{bmatrix} U_\infty^2 & 0 & -1 \\ dx & dt & 0 \\ 0 & dx & dt \end{bmatrix} \begin{bmatrix} T_{xx} \\ T_{xt} \\ T_{tt} \end{bmatrix} = \begin{bmatrix} 0 \\ dT_x \\ dT_t \end{bmatrix}$$

Since it is possible to have discontinuities in the second-order derivatives of the dependent variable across the characteristics, these derivatives are indeterminate. Therefore,

$$\begin{vmatrix} U_\infty^2 & 0 & -1 \\ dx & dt & 0 \\ 0 & dx & dt \end{vmatrix} = 0$$

or, $$\left(\frac{dx}{dt}\right)^2 - U_\infty^2 = 0$$

Solving the equation yields the equations of the characteristics

$$\frac{dx}{dt} = \pm U_\infty$$

Therefore , there are two real characteristics of the governing equation and hence the given partial differential equation is hyperbolic in nature.