Discretization- Finite difference, Finite element methods

Q1.
Identify the natural and essential boundary conditions of the following differential equation:

$\frac{d^2}{dx^2}\left[a(x)\frac{d^2y}{dx^2}\right]+b(x)=0$, for $0<x<L$; subject to the following boundary conditions:

$y=0$ and $dy/dx=0$ at $x=0$; $\left[a(x)\frac{d^2y}{dx^2}\right]_{x=L}=A$, $\left[\frac{d}{dx}\left(a(x)\frac{d^2y}{dx^2}\right)\right]_{x=L}=0$.

Solution

$$\int\left[\frac{d^2}{dx^2}\left\{a(x)\frac{d^2y}{dx^2}\right\}+b(x)\right]vdx=0$$

$$\int_0^L\frac{d^2}{dx^2}\left\{a(x)\frac{d^2y}{dx^2}\right\}vdx+\int_0^L b(x)vdx=0$$

$$\left[v\frac{d}{dx}\left\{a(x)\frac{d^2y}{dx^2}\right\}\right]_0^L-\int_0^L\frac{dv}{dx}\frac{d}{dx}\left\{a(x)\frac{d^2y}{dx^2}\right\}dx+\int_0^L b(x)vdx=0$$

$$\left[v\frac{d}{dx}\left\{a(x)\frac{d^2y}{dx^2}\right\}\right]_0^L-\left[\frac{dv}{dx}a(x)\frac{d^2y}{dx^2}\right]_0^L+\int_0^L\frac{d^2v}{dx^2}a(x)\frac{d^2y}{dx^2}dx+\int_0^L b(x)vdx=0$$

It is clear from the boundary terms that the primary variables are y and $\frac{dy}{dx}$ and secondary variables are $\frac{d}{dx}\left\{a(x)\frac{d^2y}{dx^2}\right\}$ and $a(x)\frac{d^2y}{dx^2}$ respectively. Hence, essential boundary conditions are $y=0$ and $dy/dx=0$ at $x=0$ and natural boundary conditions are $\left[a(x)\frac{d^2y}{dx^2}\right]_{x=L}=A$, $\left[\frac{d}{dx}\left(a(x)\frac{d^2y}{dx^2}\right)\right]_{x=L}=0$.

Q2.
Consider the following differential equation on the domain $x_1\le x\le x_2$

$\frac{d^2}{dx^2}\left(a\frac{d^2u}{dx^2}\right)-\frac{d}{dx}\left(b\frac{du}{dx}\right)+cu=f$, where a,b,c are known functions of x, $a(x),b(x),c(x)\ge 0$ for all x.

(a) Develop the variational formulation for this differential equation in the form: Find u such that $a(u,v)=L(v)$ for all v

(b) Identify the function spaces that u and v should lie in, and the appropriate essential and natural boundary conditions.

(c) Show that $a(.,.)$ is symmetric and positive definite.

(d) Formulate the minimization problem corresponding to the above variational formulation.

Solution

(a)

$$\int_{x_1}^{x_2} \frac{d^2}{dx^2}\left(a\frac{d^2u}{dx^2}\right)vdx - \int_{x_1}^{x_2} \frac{d}{dx}\left(b\frac{du}{dx}\right)vdx + \int_{x_1}^{x_2} cuvdx = \int_{x_1}^{x_2} fvdx$$

$$\left[v\frac{d}{dx}\left(a\frac{d^2u}{dx^2}\right)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\left(a\frac{d^2u}{dx^2}\right)\frac{dv}{dx}dx - \left[vb\frac{du}{dx}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} b\frac{du}{dx}\frac{dv}{dx}dx + \int_{x_1}^{x_2} cuvdx = \int_{x_1}^{x_2} fvdx$$

$$\left[v\frac{d}{dx}\left(a\frac{d^2u}{dx^2}\right)\right]_{x_1}^{x_2} - \left[\frac{dv}{dx}a\frac{d^2u}{dx^2}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} a\frac{d^2u}{dx^2}\frac{d^2v}{dx^2}dx - \left[vb\frac{du}{dx}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2} b\frac{du}{dx}\frac{dv}{dx}dx + \int_{x_1}^{x_2} cuvdx = \int_{x_1}^{x_2} fvdx$$

$$a(u,v) = L(v)$$

$$a(u,v) = \int_{x_1}^{x_2} a\frac{d^2u}{dx^2}\frac{d^2v}{dx^2}dx + \int_{x_1}^{x_2} b\frac{du}{dx}\frac{dv}{dx}dx + \int_{x_1}^{x_2} cuvdx$$

$$L(v) = \int_{x_1}^{x_2} fvdx - \left[v\frac{d}{dx}\left(a\frac{d^2u}{dx^2}\right)\right]_{x_1}^{x_2} + \left[\frac{dv}{dx}a\frac{d^2u}{dx^2}\right]_{x_1}^{x_2} + \left[vb\frac{du}{dx}\right]_{x_1}^{x_2}$$

(b) $\int_{x_1}^{x_2}\left(\frac{du}{dx}\right)^2 dx < \infty \to H^1$ $\quad\quad$ $\int_{x_1}^{x_2}\left(\frac{d^2u}{dx^2}\right)^2 dx < \infty \to H^2$

$\int_{x_1}^{x_2}\left(\frac{dv}{dx}\right)^2 dx < \infty \to H^1$ $\quad\quad$ $\int_{x_1}^{x_2}\left(\frac{d^2v}{dx^2}\right)^2 dx < \infty \to H^2$

(c) Possible boundary conditions are as follows

	Essential boundary conditions	Natural boundary conditions
(i)	u specified	$\frac{d}{dx}\left(a\frac{d^2u}{dx^2}\right)$ specified
(ii)	u specified	$a\frac{d^2u}{dx^2}$ specified
(iii)	u specified	$b\frac{du}{dx}$ specified

(c) $a(u,v) = a(v,u)$

It implies that a is symmetric.

$$a(v,v) = \int_{x_1}^{x_2} a\left(\frac{d^2v}{dx^2}\right)^2 dx + \int_{x_1}^{x_2} b\left(\frac{dv}{dx}\right)^2 dx + \int_{x_1}^{x_2} cv^2dx \geq 0$$

Therefore, a is positive.

(d) $\pi = \frac{1}{2}a(u,u) - L(u)$

$$=\frac{1}{2}\left[\int_{x_1}^{x_2} a\left(\frac{d^2u}{dx^2}\right)^2 dx+\int_{x_1}^{x_2} b\left(\frac{du}{dx}\right)^2 dx+\int_{x_1}^{x_2} cu^2 dx\right]$$

$$-\int_{x_1}^{x_2} fudx-\left[u\frac{d}{dx}\left(a\frac{d^2u}{dx^2}\right)\right]_{x_1}^{x_2}+\left[\frac{du}{dx}a\frac{d^2u}{dx^2}\right]_{x_1}^{x_2}+\left[ub\frac{du}{dx}\right]_{x_1}^{x_2}$$

Q3.

Consider the following heat conduction problem $(0 \le x \le 1)$:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+S=0$$

The boundary conditions specified are as follows: $T(1)=\sqrt{2}$, $\left(\frac{dT}{dx}\right)_{x=0}=0$

(i) Is $T=\cos(\pi x)+\sec(\frac{\pi x}{4})$ a valid trial function? Explain.

(ii) Is $T=\sin(\pi x)+\cos ec(\pi x)$ a valid weighting function? Explain.

Solution

(i)

$$T=\cos(\pi x)+\sec(\frac{\pi x}{4})$$

$$T(1)=\cos(\pi)+\sec(\frac{\pi}{4})=-1+\sqrt{2}$$

It violates the given essential boundary condition $\left(T(1)=\sqrt{2}\right)$. Hence $T=\cos(\pi x)+\sec(\frac{\pi x}{4})$ is not a valid trial function.

(ii)

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+S=0$$

$$\int\left[\frac{d}{dx}\left(k\frac{dT}{dx}\right)+S\right]vdx=0$$

where v is the variation in T which is taken as weighting function (w).

$$w=\delta T$$

Since T is given at $x=1$, $\delta T=0$ at $x=1$.

Now, $w=\sin(\pi x)+\cos ec(\pi x)$

$$w(1)=\sin(\pi)+\cos ec(\pi)=\text{undefined}$$

Hence $T=\sin(\pi x)+\cos ec(\pi x)$ is not a valid weighting function.

Q4.

One-dimensional steady fully developed fluid flow takes place between two parallel plates (this effectively implies that the transient and advection terms in the momentum equation turn out to be identically zero) with zero pressure gradient. However, the flow is subjected to other body forces dependent on the position and velocity, so that the corresponding governing equation for velocity in a non-dimensional form becomes

$$\frac{d^2u}{dx^2} + u + x = 0, \text{ with } u(0) = u(1) = 0.$$

Considering a trial function of $u = a\sin \pi x$, determine the value of the parameter a by following the least square method, the point collocation method (considering a single collocation point as the mid point of the domain) , Galerkin's Method and Rayleigh –Ritz Method

Solution

$$\frac{d^2u}{dx^2} + u + x = 0$$

$$u(0) = u(1) = 0$$

Trial function is given as $u_{approx} = a\sin \pi x$

Differentiating the trial function with respect to x twice, we get

$$\frac{du_{approx}}{dx} = a\pi \cos \pi x$$

$$\frac{d^2u_{approx}}{dx^2} = -a\pi^2 \sin \pi x$$

The residual becomes

$$R = \frac{d^2u_{approx}}{dx^2} + u_{approx} + x$$

$$= -a\pi^2 \sin \pi x + a\sin \pi x + x$$

or, $$R = x + a\sin \pi x\left(1-\pi^2\right)$$

i) Least Square Method.

$$\frac{\partial}{\partial a}\int_0^1 R^2 dx = 0$$

$$\int_0^1 R\frac{\partial R}{\partial a}dx = 0$$

$$\int_0^1 \left\{x + a\sin \pi x\left(1-\pi^2\right)\right\}\left\{\pi\left(1-\pi^2\right)\sin \pi x\right\}dx = 0$$

$$\Rightarrow \int_0^1 \left\{x + a\sin \pi x\left(1-\pi^2\right)\sin \pi x\right\}dx = 0$$

$$\Rightarrow \int_0^1 x\sin \pi x dx + \left(1-\pi^2\right)a\int_0^1 \sin^2 \pi x dx = 0$$

$$\Rightarrow \left[\frac{x\cos \pi x}{\pi}\right]_0^1 + \left[\frac{\sin \pi x}{\pi^2}\right]_0^1 + \left(1-\pi^2\right)a\left(\frac{1}{2}\right) = 0$$

$$\Rightarrow \frac{1}{\pi} + \left(1-\pi^2\right)\frac{a}{2} = 0$$

$$\Rightarrow \quad a = \frac{2}{\pi\left(\pi^2-1\right)}$$

$$u = \frac{2}{\pi\left(\pi^2-1\right)}\sin \pi x$$

ii) Point collocation Method :

$$R_{x=0.5} = 0$$

$$R = \frac{1}{2} + a\sin\frac{\pi}{2}x\left(1-\pi^2\right) = 0$$

$$\Rightarrow a = \frac{1}{\pi^2-1}$$

$$u = \frac{1}{\pi^2-1}\sin \pi x$$

iii) Galerkin's Method

According to Galerkins Method, weighting function is equal to trial function. Therefore, Weighting function $w = \sin \pi x$

$$\int_0^1 \left\{x + a\sin \pi x\left(1-\pi^2\right)\right\}\sin \pi x dx$$

$$a\left(1-\pi^2\right)\int_0^1 \sin^2 \pi x + \int_0^1 x\sin \pi x dx = 0$$

$$a = \frac{-\int_0^1 x\sin \pi x dx}{\left(1-\pi^2\right)\int_0^1 \sin^2 \pi x}$$

$$= \frac{2}{\pi\left(\pi^2-1\right)}$$

$$u = \frac{2}{\pi\left(\pi^2-1\right)}\sin \pi x$$

iv) Rayleigh –Ritz Method

$$\int_0^1 \left(\frac{\partial^2 u}{dx^2} + u + x\right)v dx = 0$$

$$\Rightarrow \int_0^1 \frac{\partial^2 u}{dx^2} v dx + \int_0^1 uvdx + \int_0^1 vxdx = 0$$

$$\Rightarrow \left[\frac{du}{dx} v \right]_0^1 - \int_0^1 \frac{dv}{dx}\frac{du}{dx} dx - \int_0^1 uvdx + \int_0^1 vxdx = 0$$

$$\Rightarrow \int_0^1 \left(\frac{du}{dx}\frac{dv}{dx} - uv \right) dx = \int_0^1 vxdx$$

$$a(u,v) = l(v)$$

$$\pi = \frac{1}{2} a(u,u) - l(u)$$

$$= \frac{1}{2} \int_0^1 \left(\left(\frac{du}{dx} \right)^2 - u^2 \right) dx - \int_0^1 uxdx$$

$$\pi = \frac{1}{2} \int_0^1 \left(a^2 \pi^2 \cos^2 \pi x - a^2 \sin^2 \pi x \right) dx - \int_0^1 xa \sin \pi x dx$$

$$\pi = \frac{a^2 \pi^2}{2} \times \frac{1}{2} - \frac{a^2}{2} \times \frac{1}{2} - \frac{a}{\pi}$$

$$\Rightarrow \frac{\partial \pi}{\partial a} = 0$$

$$\Rightarrow \frac{2a\pi^2}{4} - \frac{a}{2} - \frac{a}{\pi} = 0$$

$$\Rightarrow a = \frac{2}{\pi(\pi^2 - 1)}$$

$$u = \frac{2}{\pi(\pi^2 - 1)} \sin \pi x$$