Stability Analysis

Q1.

Consider the solution of the following template 1 –D wave equation: $\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$

Using a modified FTCS scheme, in which the term u_i^n for time discretization is expressed as $u_i^u=\frac{1}{2}\left(u_{i+1}^u+u_{i-1}^u\right)$, where the index ‘i’ represents spatial discretization where as the superscript ‘n’ represents temporal discretization. Examine the numerical stability of this scheme using von- Neumann stability analysis.

Solution

$$\frac{\partial u}{\partial t}+C\frac{\partial u}{\partial x}=0$$

Discretizatizing using FTCS scheme,

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}+C\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}=0$$

$$\frac{u_i^{n+1}-\frac{1}{2}\left(u_{i+1}^n+u_{i-1}^n\right)}{\Delta t}+C\frac{u_{i+1}^n-u_{i-1}^n}{2\Delta x}=0 \text{ (Given that } u_i^u=\frac{1}{2}\left(u_{i+1}^u+u_{i-1}^u\right),)$$

or, $$u_i^{n+1}-\frac{1}{2}\left(u_{i+1}^n+u_{i-1}^n\right)+\frac{Co}{2}\left(u_{i+1}^n-u_{i-1}^n\right)=0 \text{(where } Co=\frac{C\Delta t}{\Delta x}\text{)}$$

or, $$u_i^{n+1}=u_{i+1}^n\left(\frac{1}{2}-\frac{\text{Co}}{2}\right)+u_{i-1}^n\left(\frac{1}{2}+\frac{\text{Co}}{2}\right)$$

Let $u(x,t)=e^{\alpha t}e^{jkx}$

The above equation becomes

$$e^{\alpha(t+\Delta t)}e^{jkx}=e^{\alpha t}e^{jk(x+\Delta x)}\left(\frac{1}{2}-\frac{\text{Co}}{2}\right)+e^{\alpha t}e^{jk(x-\Delta x)}\left(\frac{1}{2}+\frac{\text{Co}}{2}\right)$$

or, $$\frac{e^{\alpha(t+\Delta t)}}{e^{\alpha t}}=e^{jk\Delta x}\left(\frac{1}{2}-\frac{\text{Co}}{2}\right)+e^{-jk\Delta x}\left(\frac{1}{2}+\frac{\text{Co}}{2}\right)$$

or, $$\frac{e^{\alpha(t+\Delta t)}}{e^{\alpha t}}=e^{j\theta}\left(\frac{1}{2}-\frac{\text{Co}}{2}\right)+e^{-j\theta}\left(\frac{1}{2}+\frac{\text{Co}}{2}\right) \text{ where } \theta=k\Delta x$$

or, $$\frac{e^{\alpha(t+\Delta t)}}{e^{\alpha t}}=(\cos\theta+j\sin\theta)\left(\frac{1}{2}-\frac{\text{Co}}{2}\right)+(\cos\theta-j\sin\theta)\left(\frac{1}{2}+\frac{\text{Co}}{2}\right)=A \text{ (say)}$$

Thus, $A=\cos\theta-\text{Co}j\sin\theta$

For stability,

$$|A|=\sqrt{\cos^2\theta+\text{Co}^2\sin^2\theta}\le 1$$

or, $\cos^2\theta+\text{Co}^2\sin^2\theta\le 1$

or, $\cos^2\theta+\text{Co}^2\sin^2\theta\le\cos^2\theta+\sin^2\theta$

or, $(\text{Co}^2-1)\sin^2\theta\le 1$

or, $\text{Co}^2\le 1$

or, $\text{Co}\le 1$