Convection-diffusion problems

Q1.

For a 1-D convection – diffusion problem, fluid density = 1000kg/m^3, flow velocity = 1m/s, diffusion coefficient = 10^{-9} m^2/s, and domain length = 1m. Will a central difference scheme work, for a numerical solution of this problem (Given that dimension of the solution vector for the TDMA should not exceed 1000)? Give reasons for your answer?

Solution

Cell size $\approx \frac{1}{1000}$ m

Cell $Pe \approx \frac{\rho u}{\frac{\Gamma}{\Delta x}} \approx \frac{10^3 \times 1 \times 10^{-3}}{10^{-9}} \approx 10^9 \left(\gg 1\right)$

Central difference scheme (CDS) will not work because CDS is suitable for $\left|\text{Cell } Pe\right| < 2$

Q2.

The temperature variation in condenser tube is given by $\dot{m}C\frac{dT}{dx} = \frac{UA}{L}\left(T_0 - T\right)$, where $\dot{m}$ is the mass flow rate, C is the specific heat, T is the temperature of cooling water, T_0 is the constant temperature of the condensing steam, U is the overall heat transfer coefficient, A and is the total heat transfer area. Define a non-dimensional temperature $\theta = \frac{T - T_{in}}{T_0 - T_{in}}$, $y = \frac{x}{L}$, Obtain θ as a function of y numerically, taking only 5 grid points using upwind scheme. Also compare with the exact solution. You may take $\frac{AU}{\dot{m}C} = 2$

Solution

$$\frac{dT}{dx} = \frac{dT}{d\theta}\frac{d\theta}{dy}\frac{dy}{dx} = \frac{\left(T_0 - T_{in}\right)}{L}\frac{d\theta}{dy}$$

$$1 - \theta = 1 - \frac{T - T_{in}}{T_0 - T_{in}} = \frac{T_0 - T}{T_0 - T_{in}}$$

$$T_0 - T = \left(T_0 - T_{in}\right)\left(1 - \theta\right)$$

Governing differential equation becomes

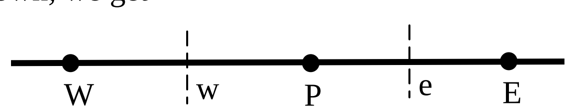
$$\dot{m}C\frac{\left(T_0 - T_{in}\right)}{L}\frac{d\theta}{dy} = \frac{UA}{L}\left(T_0 - T_{in}\right)\left(1 - \theta\right)$$

or, $$\frac{d\theta}{dy} = \frac{UA}{\dot{m}C}\left(1 - \theta\right) = 2\left(1 - \theta\right)$$

or, $$\frac{d\theta}{dy} = \frac{UA}{\dot{m}C}\left(1 - \theta\right) = 2\left(1 - \theta\right)$$

Integrating for an elemental control volume as shown, we get

$$\int_w^e \frac{d\theta}{dy}dy = \int_w^e 2\left(1 - \theta\right)dy$$



Assuming piecewise constant profile for θ

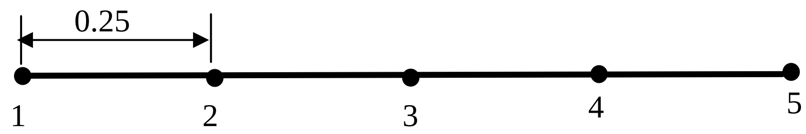
$$\theta_e - \theta_w = 2\left(1-\theta_P\right)\int_w^e dy$$

Using upwind scheme

$$\theta_P - \theta_W = 2\left(1-\theta_P\right)\Delta y$$

Comparing with the equation $a_P\theta_P = a_E\theta_E + a_W\theta_W + b$, we get

$a_P = 1+2\Delta y = 1.5$, $a_E = 0$, $a_W = 1$, $b = 2\Delta y = 0.5$



From the definition of θ, $\theta_1 = 0$

$$1.5\theta_2 = \theta_1 + 0.5$$
$$1.5\theta_3 = \theta_2 + 0.5$$
$$1.5\theta_4 = \theta_3 + 0.5$$
$$1.5\theta_5 = \theta_4 + 0.5$$

After solving above equations

$$\theta_2 = 0.333$$
$$\theta_3 = 0.555$$
$$\theta_4 = 0.703$$
$$\theta_5 = 0.802$$

Exact solution can be found as follows:

$$\frac{d\theta}{dy} + 2\theta = 2$$

or, $$e^{2y}\frac{d\theta}{dy} + 2e^{2y}\theta = 2e^{2y}$$

or, $$\frac{d}{dy}\left(\theta e^{2y}\right) = 2e^{2y}$$

or, $$\theta e^{2y} = e^{2y} + C$$

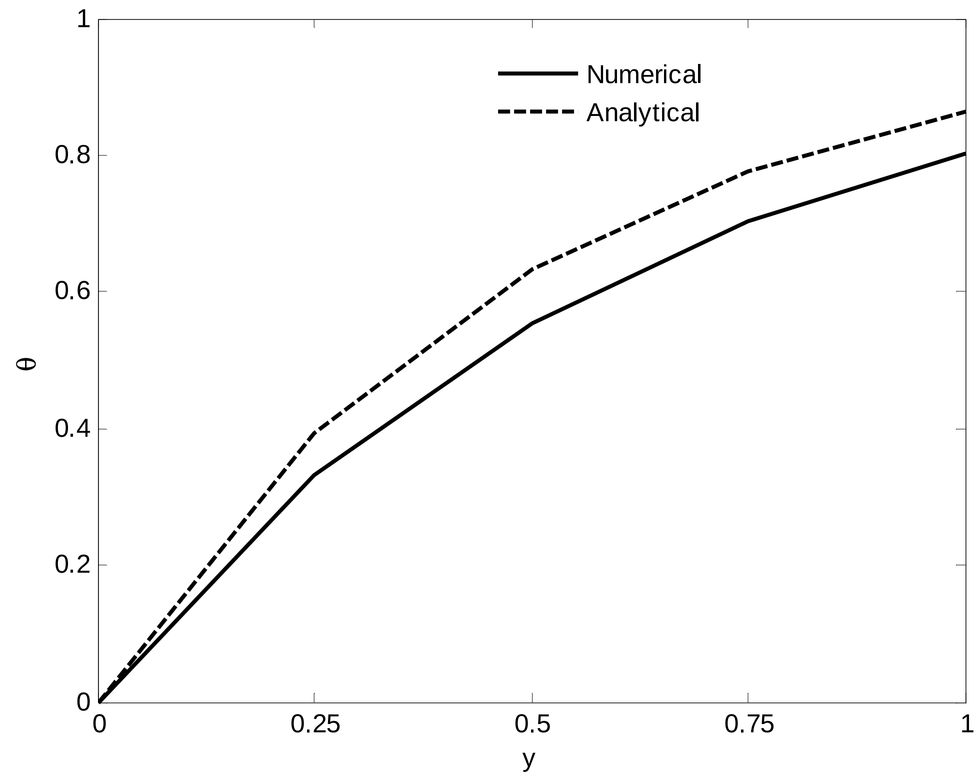
At $y = 0$, $\theta = 0$

$$0 = 1 + C$$
$$\Rightarrow C = -1$$
$$\theta = 1 - e^{-2y}$$

From the definition of θ, $\theta_1 = 0$

Putting the values of y, we get

$\theta_2 = 0.393$, $\theta_3 = 0.632$, $\theta_4 = 0.777$, $\theta_5 = 0.865$



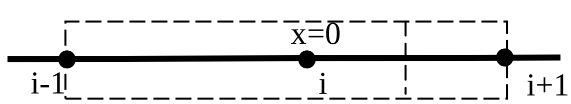
Comparison of numerical solution with analytical solution

Q3.

Consider a 1-D steady state convection – diffusion problem without any source term. Derive a profile assumption for variation of the dependent variable in the advection term, following the QUICK scheme. ased on that, derive the complete disctretization equation for the convection-diffusion problem. Assess your discretization in perspective of the basic rule regarding the sign of coefficients of the discretized equation.

(b) Extend your derivations made in part (a) to a 1 – D unsteady state convection-diffusion problem with fully explicit time discretization.

Solution



$$\phi = a_0 + a_1 x + a_2 x^2$$

$$\phi_{i-1} = a_0 + a_1 \left(-\Delta x\right) + a_2 \left(\Delta x\right)^2 \quad (12)$$

$$\phi_i = a_0 \quad (13)$$

$$\phi_{i+1} = a_0 + a_1 \Delta x + a_2 \left(\Delta x\right)^2 \quad (14)$$

Subtracting Eq, (14) from Eq. (12) , we get

$$2a_1\Delta x = \phi_{i+1} - \phi_{i-1}$$

or, $$a_1 = \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}$$

Adding Eq, (12) and Eq. (14), we have

$$2a_0 + 2a_2\left(\Delta x\right)^2 = \phi_{i+1} + \phi_{i-1}$$

or, $$a_2 = \frac{\phi_{i+1} + \phi_{i-1} - 2\phi_i}{2\left(\Delta x\right)^2}$$

$$\phi_{\text{int}} = a_0 + a_1\frac{\Delta x}{2} + a_2\frac{\left(\Delta x\right)^2}{4}$$

$$= \phi_i + \frac{\phi_{i+1} - \phi_{i-1}}{4} + \frac{\phi_{i+1} + \phi_{i-1} - 2\phi_i}{8}$$

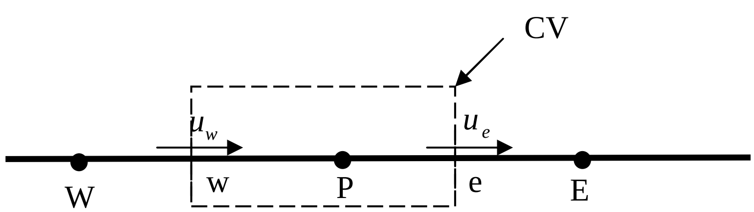
$$= \frac{6\phi_i + 3\phi_{i+1} - \phi_{i-1}}{8}$$

One-dimensional steady state convection – diffusion problem without any source term can be expressed as

$$\frac{d}{dx}\left(\rho u\phi\right) = \frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)$$

Integrating with respect to control volume as shown below, we get

$$F_e\phi_e - F_w\phi_w = \Gamma_e\left.\frac{d\phi}{dx}\right|_e - \Gamma_w\left.\frac{d\phi}{dx}\right|_w \tag{15}$$



$$F_e\phi_e - F_w\phi_w = F_e\left(\frac{6\phi_P + 3\phi_E - \phi_W}{8}\right) - F_w\left(\frac{6\phi_W + 3\phi_P - \phi_{WW}}{8}\right)$$

$$\Gamma_e\left.\frac{d\phi}{dx}\right|_e - \Gamma_w\left.\frac{d\phi}{dx}\right|_w = \Gamma_e\left.\left(a_1 + 2a_2\frac{\Delta x}{2}\right)\right|_e - \Gamma_w\left.\left(a_1 + 2a_2\frac{\Delta x}{2}\right)\right\|_w$$

$$= \Gamma_e\left(\frac{\phi_E - \phi_W}{2\Delta x} + \frac{\phi_E + \phi_W - 2\phi_P}{2\Delta x}\right) - \Gamma_w\left(\frac{\phi_P - \phi_{WW}}{2\Delta x} + \frac{\phi_P + \phi_{WW} - 2\phi_W}{2\Delta x}\right)$$

Assuming piecewise linear ϕ profiles for the diffusion term, and substituting the same in Eq. (15), we get

$$\frac{F_e}{8}\left(6\phi_P+3\phi_E-\phi_W\right)-\frac{F_w}{8}\left(6\phi_W+3\phi_P-\phi_{WW}\right)=\frac{D_e}{2}\left(2\phi_E-2\phi_P\right)-\frac{D_w}{2}\left(2\phi_P-2\phi_W\right)$$

Comparing this equation with the standard template equation
$a_P\phi_P=a_E\phi_E+a_W\phi_W+a_{WW}\phi_{WW}$, we get

$$a_E=D_e-\frac{3F_e}{8}$$

$$a_W=D_w+\frac{F_e}{8}+6\frac{F_w}{8}$$

$$a_{WW}=-\frac{F_w}{8}$$

Since a_{WW} is negative , the scheme becomes unconditionally unstable.

(b)

$$\frac{\partial}{\partial t}(\rho\phi)+\frac{\partial}{\partial x}(\rho u\phi)=\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)$$

$$\frac{\partial}{\partial t}(\rho\phi)+\frac{\partial}{\partial x}\left(\rho u\phi-\Gamma\frac{\partial\phi}{\partial x}\right)=0$$

Integrating with respect to *t* and *x* with piecewise constant ϕ profile within a control volume at a given *t* for the temporal form

$$\int_w^e\int_t^{t+\Delta t}\frac{\partial}{\partial t}(\rho\phi)\,dtdx+\int_t^{t+\Delta t}\int_w^e\frac{\partial}{\partial x}\left(\rho u\phi-\Gamma\frac{\partial\phi}{\partial x}\right)dxdt=0$$

$$\left(\rho\phi_P-\rho\phi_P{}^0\right)\Delta x+\int_t^{t+\Delta t}\left[\left(\rho u\phi-\Gamma\frac{\partial\phi}{\partial x}\right)_e-\left(\rho u\phi-\Gamma\frac{\partial\phi}{\partial x}\right)_w\right]dt=0$$

$$\rho\left(\phi_P-\phi_P{}^0\right)\Delta x+\left[\frac{F_e}{8}\left(6\phi_P{}^0+3\phi_E{}^0-\phi_W{}^0\right)-D_e\left(\phi_E{}^0-\phi_P{}^0\right)\right]-\left[\frac{F_w}{8}\left(6\phi_W{}^0+3\phi_P{}^0-\phi_{WW}{}^0\right)-D_w\left(\phi_P{}^0-\phi_W{}^0\right)\right]=0$$

Comparing this equation with the standard template equation
$a_P\phi_P=a_E\phi_E+a_W\phi_W+a_{WW}\phi_{WW}+a_P{}^0\phi_P{}^0+b$, we get

$$a_E=a_W=a_{WW}=0$$

$$a_P=\frac{\rho\Delta x}{\Delta t}$$

$$a_P{}^0=\frac{\rho\Delta x}{\Delta t}-6\frac{F_e}{8}+D_e+3\frac{F_w}{8}-D_w$$

$$b=3\frac{F_e}{8}\phi_E{}^0+\frac{\phi_W{}^0}{8}-D_e\phi_W{}^0+6\frac{F_w}{8}\phi_W{}^0-\frac{F_w}{8}\phi_{WW}{}^0+D_w\phi_W{}^0$$

$$a_P\phi_P=a_P{}^0\phi_P{}^0+b$$