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Lec. 39

$$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\times\hat{A})$$

$$\nabla\times\left[\vec{E} + \frac{\partial \vec{A}}{\partial t}\right] = 0$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla \cancel{\phi} V$$

$$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0$$

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \frac{-\rho}{\epsilon_0}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\mu_0 \vec{J}(\vec{x}, t)$$

Fourier Transform

$$\vec{A}(\vec{x}, t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \vec{A}(\vec{x}, \omega)\, e^{i\omega t}\, d\omega$$

$$\vec{A}(\vec{x}, \omega) = \int_{-\infty}^{+\infty} \vec{A}(x, t)\, e^{-i\omega t}\, dt\, .$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left[\nabla^2 + \frac{\omega^2}{c^2}\right]\vec{A}(\vec{x},\omega)e^{i\omega t}d\omega$$

$$= -\mu_0\vec{J}(\vec{x},t)$$

$$= -\mu_0 \cdot \frac{1}{2\pi}\int_{-\infty}^{+\infty}\vec{J}(\vec{x},\omega)e^{i\omega t}d\omega$$

$$\boxed{\left(\nabla^2 + \frac{\omega^2}{c^2}\right)\vec{A}(\vec{x},\omega) = -\mu_0\vec{J}(x,\omega).}$$

$$(\nabla^2 + k^2)\vec{A}(\vec{x},\omega) = -\mu_0 \vec{J}(\vec{x},\omega)$$

$$(\nabla^2 + k^2)\,G(\vec{x},\vec{x}') = -4\pi\,\delta^3(\vec{x}-\vec{x}')$$

$$\vec{A}(\vec{x},\omega) = \frac{\mu_0}{4\pi}\int G(\vec{x}-\vec{x}')\,\vec{J}(\vec{x}',\omega)\,d^3x'$$

$$(\nabla_x^2 + k^2)\vec{A}(\vec{x},\omega) = \frac{\mu_0}{4\pi}\int [(\nabla_x^2 + k^2)\,G(x-x')] \times \vec{J}(\vec{x}',\omega)\,d^3x'$$

$$= \frac{\mu_0}{4\pi}\int (-4\pi)\,\delta^3(\vec{x}-\vec{x}')\,\vec{J}(\vec{x}',\vec{\omega})\,d^3x'$$

$$= -\mu_0 \vec{J}(\vec{x},\omega)$$

$$(\nabla^2 + k^2)\, G(\vec{x} - \vec{x}\,') = -4\pi\,\delta^3(\vec{x} - \vec{x}\,')$$

$$\vec{A}(\vec{x}, \omega) = \frac{\mu_0}{4\pi} \int G(\vec{x} - \vec{x}\,')\, \vec{J}(\vec{x}\,', \omega)\, d^3x'$$

$$R = |\hat{x} - \hat{x}\,'|$$

$R \neq 0$

$$(\nabla^2 + k^2)\, G(R) = 0$$

$$\frac{1}{R} \frac{d^2}{dR^2}(RG) + k^2 G = 0$$

$$\frac{d^2}{dR^2}(RG) + k^2 (RG) = 0$$

$$RG = A\, e^{\pm ikR}$$

$$G = A \cdot \frac{e^{ikR}}{R} + B\, \frac{e^{-ikR}}{R} \qquad (R \neq 0)$$

$$\int_{R_0} (\nabla^2 + k^2) G \, d^3R = -4\pi \int_{R_0} \delta^3(R)\, d^3R = -4\pi$$

$$\int k^2 G \, d^3R = \int k^2 \cdot \left(\frac{A}{R} + \frac{B}{R} \right) d^3R$$

$$= k^2 \cdot 4\pi \, (A+B) \underbrace{\int_0^{R_0} R\, dR}_{\frac{R_0^2}{2}}$$

$$\rightarrow 0$$

$$\int_{R_0} \nabla^2 G \, d^3R = -4\pi$$

As $R_0 \to 0$ $\quad G = \frac{A+B}{R}.$

$$-4\pi(A+B) = -4\pi \qquad \nabla^2 \frac{1}{R} = -4\pi \, \delta^3(R)$$

$$A+B = 1.$$

$$G = A\frac{e^{ikR}}{R} + B\frac{e^{-ikR}}{R} \; ; \quad A+B=1$$

$$B=0$$

$$\vec{A}(\vec{x},\omega) = \frac{\mu_0}{4\pi}\int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\vec{J}(\vec{x}',\omega)\,d\omega$$

$$= \frac{\mu_0}{4\pi}\int \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\int_{-\infty}^{+\infty}\vec{J}(x',t')\,e^{i\omega t'}\cdot dt'$$

$$\vec{A}(\vec{x},t) = \frac{1}{2\pi}\,\frac{\mu_0}{4\pi}\int \frac{d^3x'}{|x-x'|}\int dt'\,J(\vec{x}',t')\underbrace{\int e^{-i\omega t'+i\omega t+i\frac{\omega}{c}|\vec{x}-\vec{x}'|}\,d\omega}$$

$$= \frac{\mu_0}{4\pi}\int \frac{d^3x'}{|x-x'|}\int_{-\infty}^{+\infty} dt'\,J(\vec{x}',t')\,\delta\!\left(t'-t+\frac{\omega}{c}|\vec{x}-\vec{x}'|\right)$$

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int \frac{d^3x'}{x-x'}\left[\vec{J}\left(x', t - \frac{|\vec{x}-\vec{x}'|}{c}\right)\right].$$

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega}$$

1. Near Fields

$$d \ll r \ll \lambda$$

2. Intermediate

$$d \ll r \approx \lambda$$

3. Far Field (Radiation Zone)

$$d \ll \lambda \ll r \, .$$

$$R = |\vec{x} - \vec{x}'| \qquad \lambda = \frac{2\pi c}{\omega}$$

$$d \ll r \ll \lambda$$

$$kR = \frac{\omega R}{c} = \frac{2\pi R}{\lambda} \ll 1$$

$$e^{ikR} \approx 1$$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

$$d \ll \lambda \ll r$$

$$\vec{x} = \hat{n}\, x \approx \hat{n}\, r$$

$$|\vec{x} - \vec{x}\,'| = \left(r^2 + x'^2 - 2 r\, \hat{n} \cdot \vec{x}\,' \right)^{1/2}$$

$$= r \left[1 - 2\, \frac{\hat{n} \cdot x'}{r} + \frac{x'^2}{r^2} \right]^{1/2}$$

$$\approx r - \hat{n} \cdot x' + \ldots.$$

$$\frac{1}{|\vec{x} - \vec{x}\,'|} = \left(r^2 + x'^2 - 2 r\, \hat{n} \cdot \vec{x}\,' \right)^{-1/2}$$