FAQs & their solutions for Module 7:
Bra-Ket Algebra and LHO-II

Question1: If $\alpha|A\rangle = |P\rangle$, show that $\langle P| = \langle A|\bar{\alpha}$ where $\bar{\alpha}$ is the adjoint of the operator α.

Solution1:

$\langle A|\bar{\alpha}|B\rangle = \overline{\langle B|\alpha|A\rangle} = \overline{\langle B|P\rangle}$

$= \langle P|B\rangle$

Since the above equation is valid for arbitrary $|B\rangle$ we have

$$\langle P| = \langle A|\bar{\alpha} = \text{conjugate of } \alpha|A\rangle \quad (1)$$

Question2: Show that $\overline{\alpha\beta} = \bar{\beta}\bar{\alpha}$ where $\bar{\alpha}$ and $\bar{\beta}$ are the adjoints of the operators α and β.

Solution2: We consider two linear operators α and β whose adjoints are denoted by $\bar{\alpha}$ and $\bar{\beta}$, respectively. Let

$$|P\rangle = \alpha\beta|A\rangle$$

then

$$\langle P| = \langle A|\overline{\alpha\beta}$$

Further, if $|Q\rangle = \beta|A\rangle$, then $|P\rangle = \alpha|Q\rangle$ and

$$\langle P| = \langle Q|\bar{\alpha} = \langle A|\bar{\beta}\bar{\alpha}$$

Thus

$$\overline{\alpha\beta} = \bar{\beta}\,\bar{\alpha} \quad (2)$$

and, in general,

$$\overline{\alpha\beta\gamma\cdots} = \cdots\bar{\gamma}\bar{\beta}\bar{\alpha} \quad (3)$$

Question3: We consider the linear harmonic oscillator problem for which

$$H = \frac{p^2}{2\mu} + \frac{1}{2}\mu\omega^2 x^2 \quad (4)$$

We introduce the operators

$$a = \frac{1}{(2\mu\hbar\omega)^{1/2}}(\mu\omega x + ip) \quad (5)$$

and

$$\bar{a} = \frac{1}{(2\mu\hbar\omega)^{1/2}}(\mu\omega x - ip) \qquad (6)$$

where we have assumed $H = \bar{H}, \quad p = \bar{p}, \quad x = \bar{x}$. Show that

$$aH - Ha = [a.H] = \hbar\omega a \qquad (7)$$

and

$$\bar{a}H - H\bar{a} = [\bar{a}, H] = -\hbar\omega\bar{a} \qquad (8)$$

Solution3:

$$\hbar\omega a\bar{a} = \frac{1}{2\mu}(\mu\omega x + ip)(\mu\omega x - ip)$$

$$= \frac{1}{2\mu}\left[\mu^2\omega^2x^2 + p^2 - i\mu\omega(xp - px)\right]$$

$$= H + \frac{1}{2}\hbar\omega \qquad (9)$$

where we have used the commutation relation

$$[x, p] = xp - px = i\hbar \qquad (10)$$

Similarly

$$\hbar\omega\bar{a}a = H - \frac{1}{2}\hbar\omega \qquad (11)$$

Thus

$$H = \frac{1}{2}\hbar\omega(\bar{a}a + a\bar{a}) \qquad (12)$$

and

$$a\bar{a} - \bar{a}a = [a, \bar{a}] = 1 \qquad (13)$$

From Eq. (4)

$$\hbar\omega a\bar{a}a = Ha + \frac{1}{2}\hbar\omega a \qquad (14)$$

and from Eq. (6)

$$\hbar\omega a\bar{a}a = aH - \frac{1}{2}\hbar\omega a \qquad (15)$$

Thus

$$aH - Ha = [a, H] = \hbar\omega a \qquad (16)$$

Similarly

$$\bar{a}H - H\bar{a} = [\bar{a}, H] = -\hbar\omega\bar{a} \qquad (17)$$

Question4: For the linear harmonic oscillator problem we have

$$H|n\rangle = E_n|n\rangle; \qquad E_n = \left(n+\frac{1}{2}\right)\hbar\omega; \qquad (18)$$

$$n = 0,1,2,3,\ldots$$

The eigenkets $|n\rangle$ form a complete set of orthonormal kets

$$\langle m|n\rangle = \delta_{mn} \qquad (19)$$

Further,

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad (20)$$

and

$$\bar{a}|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad (21)$$

Calculate $\langle x\rangle = \langle n|x|n\rangle$; $\langle x^2\rangle = \langle n|x^2|n\rangle$; $\langle p\rangle = \langle n|p|n\rangle$ & $\langle p^2\rangle = \langle n|p^2|n\rangle$ and also the uncertainty product $\Delta x\ \Delta p$, where $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ and $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$.

Solution4:

$$x = \left(\frac{\hbar}{2\mu\omega}\right)^{1/2}(a+\bar{a}) \qquad (22)$$

Thus

$$\langle n|x|n\rangle = \left(\frac{\hbar}{2\mu\omega}\right)^{1/2}\langle n|a+\bar{a}|n\rangle$$

$$= \left(\frac{\hbar}{2\mu\omega}\right)^{1/2}\left[\sqrt{n}\,\langle n|n-1\rangle + \sqrt{n+1}\,\langle n|n+1\rangle\right]$$

$$= 0 \qquad \text{[using the orthonormality condition]} \qquad (23)$$

$$\langle n|x^2|n\rangle = \left(\langle n|aa|n\rangle + \langle n|a\bar{a}|n\rangle + \langle n|\bar{a}a|n\rangle + \langle n|\bar{a}\bar{a}|n\rangle\right)$$

$$= \frac{\hbar}{2\mu\omega}\left[0+(n+1)+n+0\right]$$

$$= \frac{\hbar}{\mu\omega}\left(n+\frac{1}{2}\right) \qquad (24)$$

Thus

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{\hbar}{\mu\omega}\left(n+\frac{1}{2}\right)} \qquad (25)$$

Similarly

$$\langle n|p|n\rangle = 0 \tag{26}$$

and

$$\langle n|\,p^2\,|n\rangle = \mu\omega\hbar\left(n+\frac{1}{2}\right) \tag{27}$$

Thus

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\mu\omega\hbar\left(n+\frac{1}{2}\right)} \tag{28}$$

and

$$\Delta x \Delta p = \left(n+\frac{1}{2}\right)\hbar \tag{29}$$

The minimum uncertainty product $\left(=\frac{1}{2}\hbar\right)$ occurs for the ground state $(\ n\ =\ 0\ )$.

<u>Question5:</u> Coherent states are the eigenkets of the operator *a:*

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{30}$$

where a is the annihilation operator defined through Eq. (2). Expand $|\alpha\rangle$ in terms of the kets $|n\rangle$ and normalize $|\alpha\rangle$ to obtain

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{31}$$

The eigenvalue α can be an arbitrary complex number.

<u>Solution5:</u> We expand $|\alpha\rangle$ in terms of the kets $|n\rangle$

$$|\alpha\rangle = \sum_{n=0,1\ldots} C_n|n\rangle \tag{32}$$

Now

$$a|\alpha\rangle = \sum C_n|n\rangle = \sum_{n=1}^{\infty} C_n\sqrt{n}\,|n-1\rangle \tag{33}$$

Also

$$a|\alpha\rangle = \alpha|\alpha\rangle = \alpha\sum C_n|n\rangle \tag{34}$$

Thus

$$\alpha\ \left(\ C_0\,|0\rangle + C_1\,|1\rangle + \cdots\ \right) = C_1\,|0\rangle + C_2\sqrt{2}\ \ |1\rangle + C_3\sqrt{3}\ \ |2\rangle + \cdots$$

or

$$C_1 = \alpha C_0, \quad C_2 = \frac{\alpha C_1}{\sqrt{2}} = \frac{\alpha^2}{\sqrt{2}}C_0$$

$$C_3 = \alpha\frac{C_2}{\sqrt{3}} = \frac{\alpha^3}{\sqrt{3!}}C_0, \ldots$$

In general,

$$C_n = \frac{\alpha^n}{\sqrt{n!}} C_0 \qquad (35)$$

Thus

$$|\alpha\rangle = C_0 \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad (36)$$

If we normalize $|\alpha\rangle$, we would get

$$1 = \langle\alpha|\alpha\rangle = |C_0|^2 \sum_n \sum_m \frac{\alpha^n \alpha^{*m}}{\sqrt{n!}\sqrt{m!}} \delta_{nm}$$

$$= |C_0|^2 \sum_n \frac{\left(|\alpha|^2\right)^n}{n!} = |C_0|^2 \exp\left(|\alpha|^2\right)$$

or

$$C_0 = \exp\left(-\frac{1}{2}|\alpha|^2\right) \qquad (37)$$

within an arbitrary phase factor. Substituting in Eq. (24) we obtain

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad (38)$$

Notice that there is no restriction on the value of α; i.e., α can take *any* complex value.

Question6: $|\alpha\rangle$ and $|\beta\rangle$ are normalized eigenkets of a belonging to eigenvalues α and β. Evaluate $|\langle\alpha|\beta\rangle|^2$ and show that the eigenkets (belonging to different eigenvalues) are not orthogonal.

Solution6: $|\alpha\rangle$ and $|\beta\rangle$ are eigenkets of a belonging to eigenvalues α and β, then

$$|\langle\alpha|\beta\rangle|^2 = \left| \exp\left(-\frac{1}{2}|\alpha|^2\right) \exp\left(-\frac{1}{2}|\beta|^2\right) \sum_n \sum_m \frac{\alpha^{*n} \beta^m}{\sqrt{n!\ m!}} \langle n|m\rangle \right|^2$$

$$= \exp\left(-|\alpha|^2 - |\beta|^2\right) \left| \sum_n \frac{\left(\alpha^* \beta\right)^n}{n!} \right|^2$$

$$= \exp\left(-|\alpha|^2 - |\beta|^2 + \alpha^* \beta + \alpha \beta^*\right) = \exp\left(-|\alpha - \beta|^2\right) \qquad (39)$$

Thus the eigenkets are not orthogonal (this is because a is not a real operator).

Question7: Assume that at $t = 0$, the oscillator is in the coherent state

$$|\Psi(t=0)\rangle = |\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle \qquad (40)$$

What will be the time evolution of the state $|\Psi(t)\rangle$?

Solution7:

$$\left|\Psi\left(t=0\right)\right\rangle = \left|\alpha\right\rangle = \exp\left(-\frac{1}{2}\left|\alpha\right|^2\right)\sum\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle \qquad (41)$$

Since $\left|n\right\rangle$ are the eigenkets of the Hamiltonian, we will have

$$\left|\Psi\left(t\right)\right\rangle = \exp\left(-\frac{1}{2}\left|\alpha\right|^2\right)\sum\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle \exp\left[-\frac{iE_n t}{\hbar}\right] \qquad (42)$$

Since

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega; \qquad n = 0,1,2,3,\ldots \qquad (43)$$

we get

$$\left|\Psi\left(t\right)\right\rangle = \exp\left(-\frac{1}{2}\left|\alpha\right|^2\right)\sum\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle \exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right] \qquad (44)$$

Question8:

(a) In continuation of the previous problem, calculate

$$\left\langle\Psi\left(t\right)\right|x\left|\Psi\left(t\right)\right\rangle \text{ and } \left\langle\Psi\left(t\right)\right|p\left|\Psi\left(t\right)\right\rangle$$

(b) Compare the results with that of a classical oscillator.

Solution8:

We start with

$$\left|\Psi\left(t\right)\right\rangle = \exp\left(-\frac{1}{2}\left|\alpha\right|^2\right)\sum\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle \exp\left[-i\left(n+\frac{1}{2}\right)\omega t\right] \qquad (45)$$

Thus

$$\left\langle\Psi(t)\right|\bar{a}\left|\Psi(t)\right\rangle = e^{-N}\sum_m\sum_n\frac{\alpha^{*m}\alpha^n}{(m!n!)^{1/2}}e^{i(m-n)\omega t}(n+1)^{1/2}\left\langle m|n+1\right\rangle$$

$$= e^{i\omega t}e^{-N}\sum\frac{\alpha^*\left|\alpha\right|^{2n}}{n!} = \alpha^* e^{i\omega t} \qquad (46)$$

where we have used the relation $\bar{a}\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$. Similarly, using $a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$ (or, taking the complex conjugate of the above equation), we would get

$$\left\langle\Psi\left(t\right)\right|a\left|\Psi\left(t\right)\right\rangle = \alpha e^{-i\omega t} \qquad (47)$$

Assuming α to be real we get

$$\left\langle\Psi\left(t\right)\right|\bar{a}+a\left|\Psi\left(t\right)\right\rangle = 2\alpha \cos\omega t$$

Since

$$x = \left(\frac{\hbar}{2\mu\omega}\right)^{1/2} \left(a + \bar{a}\right) \qquad (48)$$

Thus

$$\langle x \rangle = \langle \Psi(t) | x | \Psi(t) \rangle = \left(\frac{\hbar}{2\mu\omega}\right)^{1/2} 2\alpha \ \cos\omega t \qquad (49)$$

or

$$\langle x \rangle = x_0 \ \cos\omega t \qquad (50)$$

where

$$x_0 = \left(\frac{\hbar}{2\mu\omega}\right)^{1/2} 2\alpha \qquad (51)$$

Similarly

$$p = i\left(\frac{\mu\hbar\omega}{2}\right)^{1/2} \left(\bar{a} - a\right) \qquad (52)$$

Thus

$$\begin{aligned} \langle p \rangle &= i\left(\frac{\mu\hbar\omega}{2}\right)^{1/2} \left[\langle \Psi(t) | \bar{a} | \Psi(t) \rangle - \langle \Psi(t) | a | \Psi(t) \rangle \right] \\ &= i\left(\frac{\mu\hbar\omega}{2}\right)^{1/2} \alpha \left[e^{i\omega t} - e^{-i\omega t} \right] \\ &= -\left(\frac{\mu\hbar\omega}{2}\right)^{1/2} 2\alpha \ \sin \ \omega t \qquad (53) \end{aligned}$$

or

$$\langle p \rangle = -\mu\omega x_0 \ \sin\omega t = \mu \frac{d\langle x \rangle}{dt} \qquad (54)$$

which represents the classical equation of motion.