Problems

The JWKB Approximation

Question 1: Assume $V(x) = \frac{1}{2}\mu\omega^2 x^2$. Calculate the values of the turning points and show that the energy eigenvalues as obtained by using the JWKB quantization condition is given by $E = E_n = (n+\frac{1}{2})\hbar\omega, n = 0, 1, 2, 3, \ldots (b)$ Plot the JWKB wave functions corresponding to $n = 5$ and $n = 7$.

Solution 1: $V(x) = \frac{1}{2}\mu\omega^2 x^2$

Thus

$$k^2(x) = \frac{2\mu}{\hbar^2}[E - \frac{1}{2}\mu\omega^2 x^2]$$

and the schrodinger equation is

$$\frac{d^2\psi}{dx^2} + k^2(x)\psi(x)$$

Turning points [where $k^2(x) = 0$] will be given by $x = \pm\sqrt{\frac{2E}{\mu\omega^2}}$

Thus $a = -\sqrt{\frac{2E}{\mu\omega^2}}$ and $b = +\sqrt{\frac{2E}{\mu\omega^2}}$

The JWKB quantization condition is

$$\begin{aligned}(n+\tfrac{1}{2})\pi &= \int_a^b k(x)\ \text{dx} \\ &= \sqrt{\tfrac{2\mu}{\hbar^2}}\int_a^b [E - \frac{1}{2}\mu\omega^2 x^2]^{\frac{1}{2}}\ \text{dx} \\ &= \sqrt{\tfrac{2\mu}{\hbar^2}}\sqrt{\tfrac{\mu w^2}{2}}\int_a^b (\alpha^2 - x^2)^{\frac{1}{2}}\ \text{dx}\end{aligned}$$

Thus

where $x = \alpha\sin\theta,\ \alpha = \sqrt{\frac{2E}{\mu\omega^2}}$

$$\begin{aligned}\int_a^b (\alpha^2 - x^2)^{\frac{1}{2}}\ \text{dx} &= \int_{-\alpha}^{+\alpha}\sqrt{\alpha^2 - x^2}\ \text{dx} \\ &= 2\int_0^{\frac{\pi}{2}}\alpha^2\cos^2\theta\ \text{d}\theta \\ &= 2\alpha^2\cdot\tfrac{1}{2}\cdot\tfrac{\pi}{2} \\ &= \tfrac{\pi}{2}\cdot\tfrac{2E}{\mu\omega^2}\end{aligned}$$

Thus $(n+\frac{1}{2})\pi = \sqrt{\frac{2\mu}{\hbar^2}}\cdot\sqrt{\frac{\mu w^2}{2}}\cdot\frac{\pi E}{\mu\omega^2}$

$\implies E = E_n = (n+\frac{1}{2})\hbar\omega;\ n = 0, 1, 2\ldots$

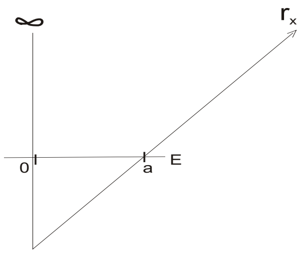
which is same as exact result.

Question 2: Assume $\begin{cases} V(x)=\infty, & for\ x<0; \\ =\gamma x, & for\ x>0. \end{cases}$ (see fig 1.)

Calculate the values of the turning points and calculate the energy eigen values as obtained by using the JWKB quantization condition.

Solution 2: For $X>0$, $V(x)=\gamma x$. Thus $k^2(x)=\frac{2\mu}{\hbar^2}[E-\gamma x]; x>0$

The turning points are $x=0$ and $x=\frac{E}{\gamma}$



Thus the JWKB quantization condition is

$$\begin{aligned} (n+\tfrac{1}{2})\pi &= \int_0^{E/\gamma} k(x)\ \mathrm{dx} \\ &= \sqrt{\frac{2\mu}{\hbar^2}} \int_0^{E/\gamma} (E-\gamma x)^{\frac{1}{2}}\ \mathrm{dx} \\ &= \frac{2}{3\gamma}\sqrt{\frac{2\mu}{\hbar^2}} E^{3/2} \end{aligned} \quad \Rightarrow E = E_n = \left(\frac{\hbar^2\gamma^2}{2\mu}\right)^{\frac{1}{3}} \left[\tfrac{3}{4}(2n+1)\pi\right]^{\frac{2}{3}};\ n=0,1,2\ldots$$

where $x=\alpha\ \sin\theta,\ \alpha=\sqrt{\frac{2E}{\mu\omega^2}}$

which represents the JWKB energy eigen values. Thus

$\xi_n = \frac{E_n}{(\hbar^2\gamma^2/2\mu)^{\frac{1}{3}}} = \left[\frac{3}{4}(2n+1)\pi\right]^{\frac{2}{3}}$

or, $\xi_n = 1.7707, 3.6838, 5.1775, \ldots$ corresponding to $n=0,1,2,\ldots$. The exact values are

$\xi_n = 2.3381, 4.0879, 5.5206, \ldots$.

Question 3: In the above problem assume the JWKB solution which vanishes at the origin and then using the condition that the solution at large values of x should be exponentially decaying , obtain the energy eigen values and compare with the exact result $E=2.3381E_0,\ 4.0879E_0,\ 5.5206E_0$ where $E_0=\left(\frac{\hbar^2\gamma^2}{2\mu}\right)13$.

Solution 3: We choose the JWKB solution which vanishes at the origin:

$$\Psi_{JWKB} = \frac{A}{\sqrt{k(x)}} \sin\left[\int_0^x k(x)\ \mathrm{dx}\right], \quad 0 < x < \frac{E}{r}$$

$$= \frac{A}{\sqrt{k(x)}} \sin\left[\theta - \left(\int_x^b + \frac{\pi}{4}\right)\right] \qquad \text{Thus}$$

$$\text{where } \theta = \int_0^b k(x)\ \mathrm{dx} + \frac{\pi}{4}$$

$$\text{and } b = \frac{E}{r}$$

$$\Psi_{JWKB} = \frac{A}{\sqrt{k}} \sin\theta \cos\left[\int_x^b k dx + \frac{\pi}{4}\right]$$

$$= \frac{-A}{\sqrt{k}} \cos\theta \sin\left[\int_x^b k dx + \frac{\pi}{4}\right]$$

The first term on the R.H.S. will go over to an exponentially amplifying solution in the region $x > b$ and therefore we must have $\sin\theta = 0$ or $\theta = n\pi$

$$\Rightarrow \int_0^b k dx = (n - \frac{1}{4})\pi; \quad n = 1, 2, \cdots$$

Thus

$$\xi_n = \frac{E_n}{(\hbar^2 r^2/2\mu)^{1/3}} = [\frac{3}{4}(2n - \frac{1}{2})\pi]^{2/3}$$

$$= 2.3203, 4.0818, 5.5172, \cdots$$

which compares well with the exact values mentioned above.

Question 4: Consider a symmetric potential energy variation as shown in the Fig.2. Assume $0 < E < V_0$

(a) Write the exponentially decaying JWKB solution in the region $x > b$.

(b) Use JWKB connection formulae to write the solution in the regions $A < x < b$ and $0 < x < a$.

(c) Use the condition $\psi(0) = 0$ to obtain the transcendental equation determining the energy eigenvalues for anti-symmetric states.

(d) Repeat the analysis for the symmetric JWKB solution.

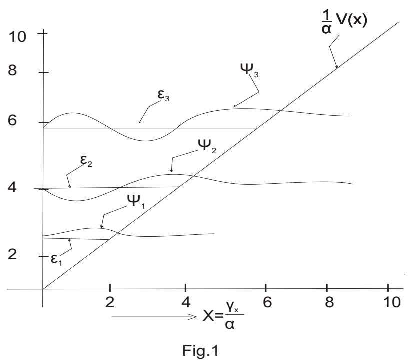


Fig.1

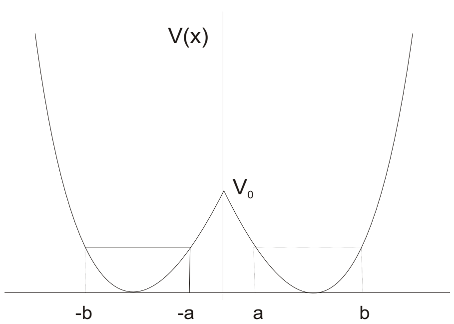


Fig 2

Solution 4: For $x > b$, the JWKB solution would be

$$\Psi(x) = \frac{A}{\sqrt{k}} exp\left[\int_b^x k(x) dx\right]; x > b$$

where $k^2(x) = \frac{2\mu}{\hbar^2}[V(x) - E]$.

The above solution would go over to

$\Psi_x = \frac{2}{\sqrt{k(x)}} \sin[\int_x^b kdx + \frac{\pi}{4}], \quad a < x < b$

Now

$\int_x^b kdx + \frac{\pi}{4} = \int_a^b kdx + \frac{\pi}{2} - (\int_a^x kdx + \frac{\pi}{4})$

Thus

$$\begin{aligned} \Psi(x) &= \frac{2}{\sqrt{k(x)}} \cos[\theta - (\int_a^x kdx + \frac{\pi}{4})] \\ &= \frac{2}{\sqrt{k(x)}}[\cos\theta \cos(\int_a^x kdx + \frac{\pi}{4}) + \sin\theta \sin(\int_a^x kdx + \frac{\pi}{4})], \ a < x < b \end{aligned}$$

$\theta = \int_a^b kdx$

We again use connection formulae to obtain

$\Psi(x) = \frac{2}{\sqrt{k(x)}}[\cos\theta e^{\int_x^a kdx} + \frac{1}{2}\sin\theta e^{-\int_x^a kdx}]$ for $-a < x < a$

The condition

$\Psi(0) = 0$

will immediately lead to

$\cot\theta = -\frac{1}{2} exp[\int_{-a}^{a} k(x)dx]$

where

$\theta = \int_a^b k(x)dx$

For the symmetric solution $\Psi'(0) = 0$ and we will get

$\cot\theta = \frac{1}{2} exp[-\int_{-a}^{a} k(x)dx]$

Question 5: In the above problem assume $V(x) = 1/2\omega^2(|x| - d)^2$ and carry out the integrations in the transcendental equations.

Solution 5: $V(x) = \frac{1}{2}\mu\omega^2(|x| - d)^2$

$V(0) = V_0 = \frac{1}{2}\mu\omega^2 d^2$

Now (for $E < V_0$)

$$\begin{aligned} \theta &= \int_a^b k(x)dx \\ &= \sqrt{\frac{2\mu}{\hbar^2}} \int_a^b [E - \frac{1}{2}\mu\omega^2(x-d)^2]^{1/2} dx \\ &= \int_{-\alpha}^{\alpha} [\alpha^2 - \xi^2]^{1/2} d\xi = \frac{\pi}{2}\alpha^2 \end{aligned}$$

where $\alpha = \sqrt{\frac{2E}{\hbar\omega}}$ and $\xi = \sqrt{\frac{\mu\omega}{\hbar}}(x - d)$

Thus

(i) For $E < V_0$

$\cot(\frac{\pi}{2}\alpha^2) = \pm\frac{1}{2}exp[-\alpha_0({\alpha_0}^2 - \alpha^2)^{1/2} + \alpha^2 \cosh^{-1}\frac{\alpha_0}{\alpha}]$

(ii) For $E > V_0$

$\frac{\pi}{2}\alpha^2 + \alpha_0(\alpha^2 - \alpha_0^2)^{1/2} + \alpha^2 \sin^{-1}\frac{\alpha_0}{\alpha} = (m + \frac{1}{2})\pi$

where $\alpha_0 = \sqrt{\frac{2V_0}{\hbar\omega}}$