PCD_STiCM_P4

Department of Physics

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STiCM: Select/Special Topics in Classical mechanics   Assignment-4

1. The potential in a region of space is given by $U(\rho,\varphi) = \rho^2 + 4\rho\cos\varphi + 5$.
   a) What must be the dimensions of the terms 'ρ^2', '4', '$\cos\varphi$' and '5'?
   b) Find the force corresponding to the above potential.
   c) If the same potential is represented in Cartesian coordinate system as U(x,y), sketch the equipotential corresponding to U(x,y)=5.
   d) In the same plot, sketch a few field lines of the force field corresponding to the given potential.

2. 'Slide-rules': The height (in meters) of a certain hill is given by

$$h(x,y) = 10\left(2xy - 3x^2 - 4y^2 - 18x + 28y + 12\right)$$

where y is the distance (in km) north, and x is the distance (in km) East of a certain town in the state of Ontario.

   a) Where is the top of the hill located?
   b) How high is the hill?
   c) How steep is the slope at a point 1 km north and 1 km east of Ontario?

3. Guess with Gauss: compute the divergence of the function,

$$\vec{V} = (r\cos\theta)\,\hat{e}_r + (r\sin\theta)\,\hat{e}_\theta + (r\sin\theta\cos\varphi)\,\hat{e}_\phi$$

Determine separately the surface integral and the volume integral that go into the Gauss's divergence theorem for the above vector point function in a region of space described by a hemisphere of radius R resting on the xy-plane, with its center at origin and located in the region $z \le 0$. Check if the result is in accordance with Gauss' law; i.e. verify that the result of the surface integral agrees with that of the volume integral.

4. The potential corresponding to a conservative force field in a region of space is given by $U(\rho,\phi,z) = A\rho z + B$.
   (i) What are the dimensions of A & B?
   (ii) Obtain the expression for the force field in the region.

5. A particle of unit mass moves in the xy-plane under the action of a force given by $\vec{F}(x,y) = -k(x\hat{e}_x - y\hat{e}_y)$.

a) What must be the dimension of k?

b) Sketch the lines of force for the force field given by $\vec{F}$.

c) Find the potential corresponding to the force and draw the equipotential surface.

6. A particle of mass m moves under the potential

$U(x,y) = -U_0 \exp[-\frac{(x^2+y^2)}{2L^2}]$, where U_0 & L are positive constants.

a) List all points of equilibria, & describe the nature of the equilibrium in each case.

b) Obtain the expression for the force $\vec{F}(x,y)$ on a particle at any point (x,y).

c) Depict on a graph sheet a few equipotential points corresponding to $U(x,y) = -\frac{U_0}{2}$

7. A force field in a region of space is given as $F = F_0(yz\hat{e}_x + zx\hat{e}_y + xy\hat{e}_z)$. Find the corresponding potential. Using divergence theorem, find the flux through any closed surface within the region.

8. A vector field is given by $\vec{A} = x^2\hat{e}_x + y^2\hat{e}_y + z^2\hat{e}_z$. Evaluate $\oiint \vec{A} \bullet \vec{ds}$ over the closed surface of a cylinder $x^2 + y^2 = 16$ bound by planes z=0,z=3

9. For the vector field described in the above problem, verify divergence theorem over a cube with $0 \le (x, y, z) \le 1$.

10. Given $\vec{A} = kr\hat{e}_r$, (k>0)

a) Determine the net flux of this vector field through the shell enclosed by two concentric spherical surfaces with radii a and b; b> a. Both the spherical surfaces are centered at origin of the coordinate frame of reference.

b) If the above vector field represents an electrostatic field, find the charge density in the region.

11. The electrostatic potential in the region 0< r< ∞ is given by

$$\phi(r,\theta,\varphi) = \frac{k}{r^2}\cos\theta .$$

a) What must be the dimension of k?
b) Find the corresponding electric field.
c) Find the volume charge density in the region.

12. A steady current density in the region of space r > 0 is given by

$$\vec{J} = J_0 e^{-\lambda r}\,\hat{e}_r \ .$$

a) What must be the dimension of J_0 & λ ?
b) Find the charge density corresponding to this current density
c) Sketch the divergence of $\vec{J}$ as a function of r.

13. A vector field representing the velocity of a fluid in motion is given as

$$\vec{V}(\rho,\phi,z) = k\vec{\nabla}\phi .$$

a) What must be the dimension of k?
b) Express the corresponding velocity in Cartesian coordinate system.
c) Sketch a few field lines for this force in all quadrants indicating the direction of flow clearly.

14. Prove that

$$(a)\ \vec{\nabla}\bullet(\vec{a}\times\vec{r}) = 0$$

$$(b)\ \vec{\nabla}\bullet(r^n\,\vec{r}) = (n+3)r^n$$

$$(c)\ \vec{\nabla}(\vec{a}\bullet(\vec{b}\times\vec{r})) = \vec{a}\times\vec{b}$$