Department of Physics
Indian Institute of technology Madras
Select/Special Topics in Classical Mechanics
Self-Assessment-3 (Questions & Answers)

Answer all the 17 (SEVENTEEN) questions STRICTLY in the space provided for the same. Extra pages are added for your rough work. ***However, do not forget to insert your final answer in the specific space designated for each question.*** **Answers written elsewhere will NOT be evaluated.**

p**1**

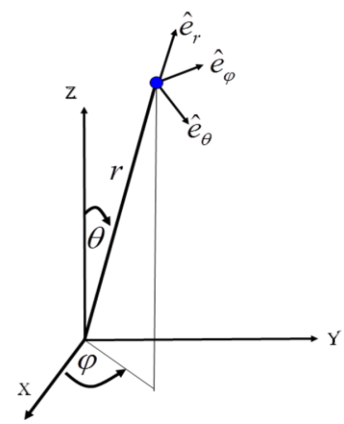
NOTE: Notations/symbols used have their usual meanings. Page numbers are given in 'circles' in left-side margin.

1. Express the quantities on the left hand sides of the equations below in terms of the coordinates and unit vectors of the spherical polar coordinate system; give your answer in the blank space on the right hand side of the equation:

$$\left[\vec{r}\right]_{\text{sph.polar coord.}} = r\hat{e}_r$$

$$\left[d\vec{r}\right]_{\text{sph.polar coord.}} = dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\varphi\,\hat{e}_\varphi$$

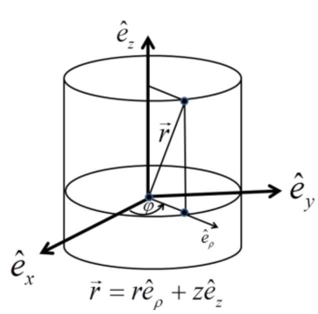
$$\left[\frac{\partial\hat{e}_\varphi}{\partial\varphi}\right]_{\text{sph.polar coord.}} = -\sin\theta\hat{e}_r - \cos\theta\hat{e}_\theta$$



2. Fill in the blank spaces in the equations below by appropriate coefficients that are functions of the cylindrical polar coordinates (ρ, φ, z):

$$-e_\rho = e_x(\_\_\_\_\_\_\_\_\_\_\_\_) + e_y(\_\_\_\_\_\_\_\_\_\_\_\_)$$

$$-e_\varphi = e_x(\_\_\_\_\_\_\_\_\_\_\_\_) + e_y(\_\_\_\_\_\_\_\_\_\_\_\_)$$



Answers: Q2 $-\hat{e}_\rho = \hat{e}_x(-\cos\varphi) + \hat{e}_y(-\sin\varphi)$ $\quad -\hat{e}_\varphi = \hat{e}_x(+\sin\varphi) + \hat{e}_y(-\cos\varphi)$

3. A vector point function is defined by $\vec{A}(\vec{r}) = 3y\hat{e}_x + 4z\hat{e}_y - 6x\hat{e}_z$.
Fill in the blanks by writing this vector in cylindrical polar coordinates.

$$\vec{A}(\vec{r}) = 3\rho\sin\varphi\left(\cos\varphi\hat{e}_\rho - \sin\varphi\hat{e}_\varphi\right) + +4z\left(\sin\varphi\hat{e}_\rho + \cos\varphi\hat{e}_\varphi\right) + -6\rho\cos\varphi\hat{e}_z$$

$\vec{A}(\vec{r}) =$ $\hat{e}_\rho$ ______________________

p**2**

$+ \hat{e}_\varphi$ ______________________

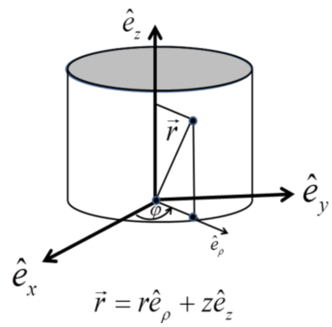
$+ \hat{e}_z$ ______________________

$$= \left(3\rho\sin\varphi\cos\varphi + 4z\sin\varphi\right)\hat{e}_\rho + \left(4z\cos\varphi - 3\rho\sin^2\varphi\right)\hat{e}_\varphi + +\left(-6\rho\cos\varphi\right)\hat{e}_z$$

4. A vector point function is defined by

$$\vec{G}(\vec{r}) = \hat{e}_\rho\{\sin\varphi + \cos\varphi\} + \hat{e}_\varphi 10\cos\varphi - 10\hat{e}_z$$

in the cylindrical polar coordinates. Find the flux of $\vec{G}$ crossing the surface of the cylinder shown in the figure from its inside to outside. The cylinder is closed at its top face, at z=2, and it is open at the bottom, which is in the XY plane at z=0.



$$\iint \vec{G}(\vec{r})\bullet\overrightarrow{dS} = \iint\left\{(\sin\varphi + \cos\varphi)\hat{e}_\rho + 10\cos\varphi\hat{e}_\varphi - 10\hat{e}_z\right\}\bullet(\rho d\varphi dz)\hat{e}_\rho + +\iint\left\{(\sin\varphi + \cos\varphi)\hat{e}_\rho + 10\cos\varphi\hat{e}_\varphi - 10\hat{e}_z\right\}\bullet(\rho d\varphi d\rho)\hat{e}_z$$

$$= -\iint 10\rho d\varphi d\rho = -\frac{10R^2}{2} \times 2\pi = -10\pi R^2$$

Write your answer to Q4 here: $-10\pi R^2$
The required FLUX is = ______________________________

p3

5. We use the usual notation: $\vec{r}$: position vector; $\vec{p}$: linear momentum; $\vec{L}$: angular momentum.

5(a) Express the triple product $\{\vec{r} \cdot (\vec{p}\times\vec{L})\}$ in terms of an appropriate function of the angular momentum $\vec{L}$ alone.

$$5(a): Answer: \{\vec{r} \cdot (\vec{p}\times\vec{L})\} = \{\vec{r} \times \vec{p} \cdot \vec{L}\} = \vec{L}\cdot\vec{L} = |\vec{L}|^2 = L^2$$

Write your answer to 5(a) here: $\{\vec{r} \cdot (\vec{p}\times\vec{L})\} = f(\vec{L}) = L^2$

5(b) Given: $\vec{V} = \vec{L}\times[\vec{p}\times\vec{L}]$. **Tick mark** ☐√ **in the box that corresponds to the correct statement(s) amongst the following:**

(i) ☑ The vector $\vec{V}$ is parallel to the linear momentum vector $\vec{p}$.

(ii) ☐ The vector $\vec{V}$ is anti-parallel to the linear momentum vector $\vec{p}$.

(iii) ☐ The vector $\vec{V}$ is orthogonal to the linear momentum vector $\vec{p}$.

(iv) ☐ The vector $\vec{V}$ makes an angle between 0^0 and 90^0 with respect to the linear momentum vector $\vec{p}$.

(v) ☐ The vector $\vec{V}$ makes an angle between 90^0 and 270^0 with respect to the linear momentum vector $\vec{p}$.

$Ans: \vec{L}\times[\vec{p}\times\vec{L}] = \vec{p}L^2 - \vec{L}(\vec{p}\cdot\vec{L}) = \vec{p}L^2$, since $\vec{p}\cdot\vec{L} = 0$. $L^2 > 0$, $\therefore$ $\vec{V}$ is parallel to $\vec{p}$ $\Rightarrow$ option (i)

5(c) Given: The Laplace-Runge-Lenz vector is $\vec{A} = (\vec{p}\times\vec{L} - mk\hat{e}_r)$, and *we* define a new vector $\vec{F} = \vec{L}\times\vec{A}$. **Tick mark** ☐√ **in the box that corresponds to the correct statement(s) amongst the following:**

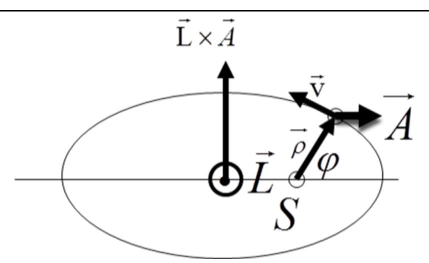
(i) ☐ The vector $\vec{F}$ is parallel to the linear momentum vector $\vec{p}$.

(ii) ☐ The vector $\vec{F}$ is parallel to the major axis of the ellipse.

(iii) ☐ The vector $\vec{F}$ orthogonal to the plane of the Kepler orbit.

(iv) ☑ The vector $\vec{F}$ is parallel to the minor axis of the ellipse.

(v) ☐ None of the above.



Answer: (iv)

Q.6 A point mass '*m*' moves along a curved trajectory in a plane at z=0 such that it is at a radial distance ρ from a centre of force, and φ is the azimuthal angle made by the position vector with
the ZX plane. <u>NOTE: Use cylindrical polar coordinate system to answer Q6.</u>

Given: The force acting on the particle is $\vec{F} = -\frac{k}{\rho^2}\hat{e}_\rho$; *k is a positive constant.*

Now, fill in the following blanks:

p**4**

(a) The work done on the particle by the force in bringing it from infinity, where the potential energy of the particle is zero, to a distance ρ from the center of the force

is : $\int_\infty^\rho \vec{F} \bullet \overrightarrow{d\rho} = \int_\infty^\rho \left(-\frac{k}{\rho^2}\hat{e}_\rho\right) \bullet \left(d\rho\hat{e}_\rho\right) = \frac{k}{\rho}$.

(b) The Lagrangian of the particle is: $L = T - V = \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2\right) + \frac{k}{\rho}$.

(c) The generalized momentum for the particle is/are: $p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}$ & $p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi}$.

(d) The Hamilton's principal function for the particle is:

$$H = \sum_i p_i \dot{q}_i - L$$

$$= p_\rho \dot{\rho} + p_\varphi \dot{\varphi} - \left\{\frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\varphi}^2\right) + \frac{k}{\rho}\right\}$$

$$= p_\rho \frac{p_\rho}{m} + p_\varphi \dot{\varphi} - \left[\frac{1}{2}m\left(\frac{p_\rho}{m}^2 + \rho^2\left\{\frac{p_\varphi}{m\rho^2}\right\}^2\right) + \frac{k}{\rho}\right].$$

NOTE: $L = L(q, \dot{q}, t)$
$H = H(q, p, t)$

$$= \frac{p_\rho{}^2}{m} + p_\varphi \frac{p_\varphi}{m\rho^2} - \frac{1}{2}\frac{p_\rho{}^2}{m} - \frac{1}{2}\frac{p_\varphi{}^2}{m\rho^2} - \frac{k}{\rho}$$

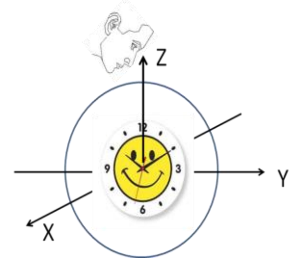
$$= \frac{1}{2}\frac{p_\rho{}^2}{m} + \frac{1}{2}\frac{p_\varphi{}^2}{m\rho^2} - \frac{k}{\rho}$$

The Hamiltonian must be written in terms of the generalized coordinates and momenta, not in terms of coordinates and velocities.

7. Find the work done by a force given by

$$\vec{F}(\vec{r}) = (3R\sin\varphi\cos\varphi + 4z\sin\varphi)\hat{e}_\rho + (4z\cos\varphi - 3R\sin^2\varphi)\hat{e}_\varphi - 6\rho\cos\varphi\hat{e}_z$$

over a circular path in the XY plane with center at the origin and radius R. The sense in which the path is traversed is clockwise as viewed from above the XY plane.



p5

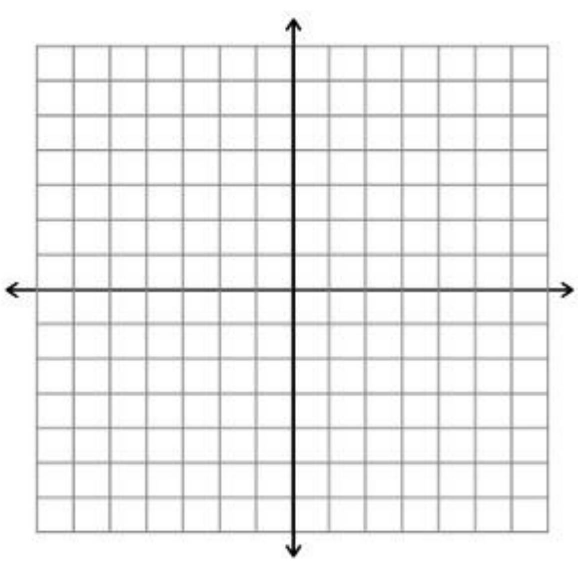
$$\oint \vec{F}(\vec{r})\cdot\vec{dr} = \oint\left[(3R\sin\varphi\cos\varphi + 4z\sin\varphi)\hat{e}_\rho + (4z\cos\varphi - 3R\sin^2\varphi)\hat{e}_\varphi - 6\rho\cos\varphi\hat{e}_z\right]\cdot\left[Rd\varphi\hat{e}_\varphi\right]$$

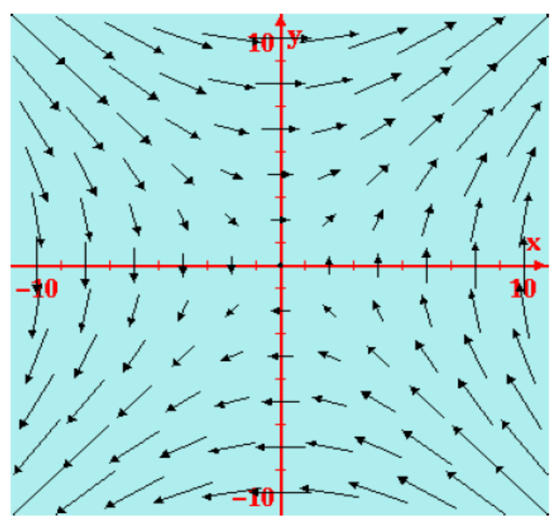
$$= -3R^2\oint\sin^2\varphi d\varphi = -3R^2\int_{\varphi=2\pi}^{0}\sin^2\varphi d\varphi = (-3R^2)\times(-\pi) = +3\pi R^2$$

Write your answer to Q7 here:
The required WORK DONE is = $+3\pi R^2$

Marks for Q7: 2

8. Sketch the vector field of the gradient of the function $\phi = 100 + xy$:





Answer: $\vec{\nabla}\phi = x\hat{e}_y + y\hat{e}_x$

9. Determine if the following statements are ‘**TRUE**’’ or ‘**FALSE**’, and **tick** √ **the right choice. Also, provide the reason:**

(a) If the fluid velocity flow is in a ‘steady state’ and it is also ‘irrotational’, then $\left\{\frac{p(\vec{r})}{\rho}+\phi+\frac{|\vec{v}|^2}{2}\right\}$ is a constant only for a particular streamline in the fluid but not for the whole velocity field.

p6

Answer here: True [ ] **False** [✓]

Reason: $-\vec{v}\times\vec{\chi}=-\vec{\nabla}\left\{\frac{p(\vec{r})}{\rho}+\varphi+\frac{|\vec{v}|^2}{2}\right\}$ *for a steady state. We also have* $\vec{\chi}=\vec{\nabla}\times\vec{v}=\vec{0}$

$\Rightarrow\left\{\frac{p(\vec{r})}{\rho}+\varphi+\frac{|\vec{v}|^2}{2}\right\}=$ *constant for the whole velocity field of the liquid* .

(b) If the velocity field of a fluid flow is in ‘steady state’ and it is also ‘solenoidal’, then the fluid is compressible.

Answer here: True [ ] **False** [✓]

$\vec{\nabla}\bullet\vec{J}(\vec{r},t)+\frac{\partial\rho(\vec{r},t)}{\partial t}=0 \qquad \frac{\partial\rho(\vec{r},t)}{\partial t}=0\Rightarrow\vec{\nabla}\bullet\{\rho(\vec{r},t)\vec{v}(\vec{r},t)\}=0$

Reason: $\therefore\ \vec{\nabla}\rho(\vec{r},t)\bullet\vec{v}(\vec{r},t)+\rho(\vec{r},t)\{\vec{\nabla}\bullet\vec{v}(\vec{r},t)\}=0$

$\vec{\nabla}\bullet\vec{v}(\vec{r},t)=0\Rightarrow\vec{\nabla}\rho(\vec{r},t)=\vec{0}\Rightarrow$ *Fluid is incompressible.*

p**7**

Q10. In the Lagrangian description of fluid flow, the velocity field of a given field is:

$\vec{u} = 2z\hat{e}_x + xt\hat{e}_y + xy^2\hat{e}_z$. For this case, find the acceleration of a fluid particle in this flow.

$$\vec{a} = \frac{d\vec{u}}{dt} = \left[\vec{u} \bullet \vec{\nabla} + \frac{\partial}{\partial t}\right]\vec{u}$$

$$a_x = \left[\vec{u} \bullet \vec{\nabla} + \frac{\partial}{\partial t}\right]2z$$

$$= \vec{u} \bullet \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}\right)$$

$$= \left(2z\hat{e}_x + xt\hat{e}_y + xy^2\hat{e}_z\right) \bullet \left(2z\hat{e}_z\right)$$

$$= 2xy^2 z$$

$$a_y = \left[\vec{u} \bullet \vec{\nabla} + \frac{\partial}{\partial t}\right]xt$$

$$= \left[\vec{u} \bullet \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}\right)xt\right] + \frac{\partial (xt)}{\partial t}$$

$$= \left(2z\hat{e}_x + xt\hat{e}_y + xy^2\hat{e}_z\right) \bullet \left(t\hat{e}_x\right) + x$$

$$= 2zt + x$$

$$a_z = \left[\vec{u} \bullet \vec{\nabla} + \frac{\partial}{\partial t}\right]xy^2$$

$$= \left[\vec{u} \bullet \left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}\right)xy^2\right]$$

$$= \left(2z\hat{e}_x + xt\hat{e}_y + xy^2\hat{e}_z\right) \bullet \left(y^2\hat{e}_x + 2xy\hat{e}_y\right)$$

$$= 2zy^2 + 2x^2yt$$

$\vec{a} = a_x\hat{e}_x + a_y\hat{e}_y + a_z\hat{e}_z$ i.e. $\vec{a} = 2xy^2\hat{e}_x + \left(2zt + x\right)\hat{e}_y + \left(2zy^2 + 2x^2yt\right)\hat{e}_z$

Write your answer to Q10 by filling in the blanks here:

$\vec{a} = 2xy^2 \quad \hat{e}_x \quad + \quad \left(2zt + x\right) \quad \hat{e}_y \quad + \quad \left(2zy^2 + 2x^2yt\right) \quad \hat{e}_z$

$\vec{a}$ = ________________ $\hat{e}_x$ + ___________________ $\hat{e}_y$ + ___________________ $\hat{e}_z$

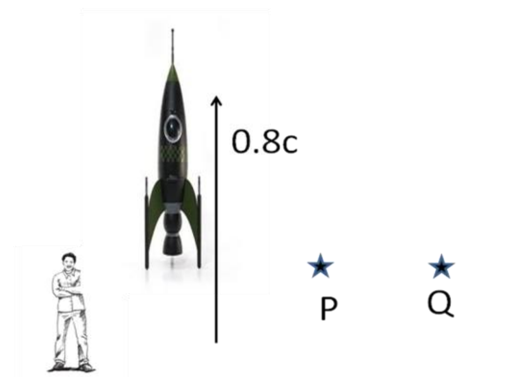
Q11. A rocket flying horizontally above Canada flying at speed of 0.8c crosses a certain region below on ground in 1.0 second as per the clock on the rocket. Find the corresponding time interval as would be clocked by an observer on the ground.

p**8**

$$T = \frac{\tau}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{(0.8c)^2}{c^2}}} = \frac{1}{\sqrt{1-0.64}} = \frac{1}{0.6} \approx 1.67s$$

Write your answer to Q11 here: The required time interval is = ≈ 1.67 **seconds.**

12. An observer in a rocket flying as shown in the figure at a speed of 0.8c measures a distance between two objects P and Q which are stationary with respect to the observer on ground to be 2ly. Find the corresponding distance as would be estimated by an observer on the ground.



Since the rocket's motion is orthogonal to PQ, it two observers record the same distance.

Write your answer to Q12 here: The required distance is = <u>2ly</u>.

p**9**

13. (a) Suppose you are standing at the equator. You throw an object directly toward the east. In which direction (if any) will it get deflected?

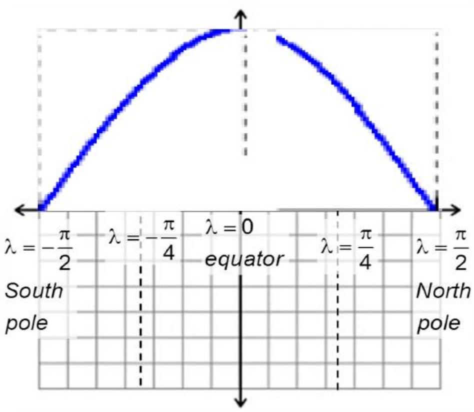
State your answer here:___________UPWARD___

(b) Suppose you are standing at the equator. You throw an object directly toward the west. In which direction (if any) will it get deflected?

State your answer here:___________DOWNWARD___

This is a special case of the Coriolis effect and is often called as the Eotvos effect.

14. Sketch the magnitude of the centrifugal acceleration of an object on earth as a function of the earth's latitude λ on the graph below:



COSINE CURVE

$$\vec{a}_{centrifugal} = -\vec{\omega} \times (\vec{\omega} \times \vec{r})$$
$$= -\omega^2 r \hat{e}_\omega \times (\hat{e}_\omega \times \hat{e}_r)$$
$$= -\omega^2 r \hat{e}_\omega \times (\sin\theta \hat{e}_\varphi)$$
$$= \omega^2 r \sin\theta \hat{e}_\rho$$
$$|\vec{a}_{centrifugal}| = \omega^2 r \sin\theta$$
$$= \omega^2 r \cos\lambda$$

15. Fill in the blanks:

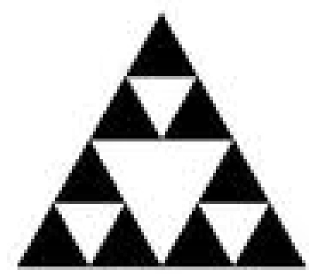
(a) The 'attractor' of simple harmonic oscillator is a limit cycle (periodic orbit) in a shape of ellipse in the phase space (q,p).

(b) The 'attractor' of a damped linear harmonic oscillator is steady state fixed point in phase space toward which the orbit would spiral in and collapse.

(c) The approximate numerical value of the 'golden ratio' (upto two places of decimal) is $1.618.... \approx 1.62$

__________________.

16. The fractal dimension d of the Sierpinski triangle is **(fill in the blank space below):**



d $= \frac{\log 3}{\log 2} \approx 1.585$