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Select / Special Topics in Classical Mechanics

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STiCM Lecture 02: Unit 1 Equations of Motion (i)

Unit 1: Equations of Motion

Equations of Motion. Principle of Causality and Newton's I & II Laws. Interpretation of Newton's 3rd Law as 'conservation of momentum' and its determination from translational symmetry.

Alternative formulation of Mechanics via 'Principle of Variation'. Determination of Physical Laws from Symmetry Principles, Symmetry and Conservation Laws.

Lagrangian/Hamiltonian formulation. Application to SHO.

Learning goals:

'Mechanical system' is described by (position, velocity) or (position, momentum) or some well defined function thereof.

How is an 'inertial frame of reference' identified?

What is the meaning of 'equilibrium'?

What causes departure from equilibrium?

Can we 'derive' I law from the II by putting

$$\left|\vec{F}\right| = 0 \text{ in } \vec{F} = m\vec{a}.$$

……. Learning goals:

Learn about the 'principle of variation' and how it provides us an alternative and more powerful approach to solve mechanical problems.

Introduction to Lagrangian and Hamiltonian methods which illustrate this relationship and apply the technique to solve the problem of the simple harmonic oscillator.

Central problem in 'Mechanics': How is the 'mechanical state' of a system described, and how does this 'state' evolve with time? Formulations due to Galileo/Newton, Lagrange and Hamilton.

Why 'position' and 'velocity' are <u>both</u> needed to specify the mechanical state of a system?

They are independent parameters that specify the 'state'.

The mechanical state of a system is characterized by its position and velocity, $(q, \dot{q})$ or, position and momentum, (q, p)

Or, equivalently by their well-defined functions:

$L(q, \dot{q})$: *Lagrangian*
$H(q, p)$: *Hamiltonian*

A two-dimensional space spanned by the two orthogonal (thus independent) degrees of freedom.
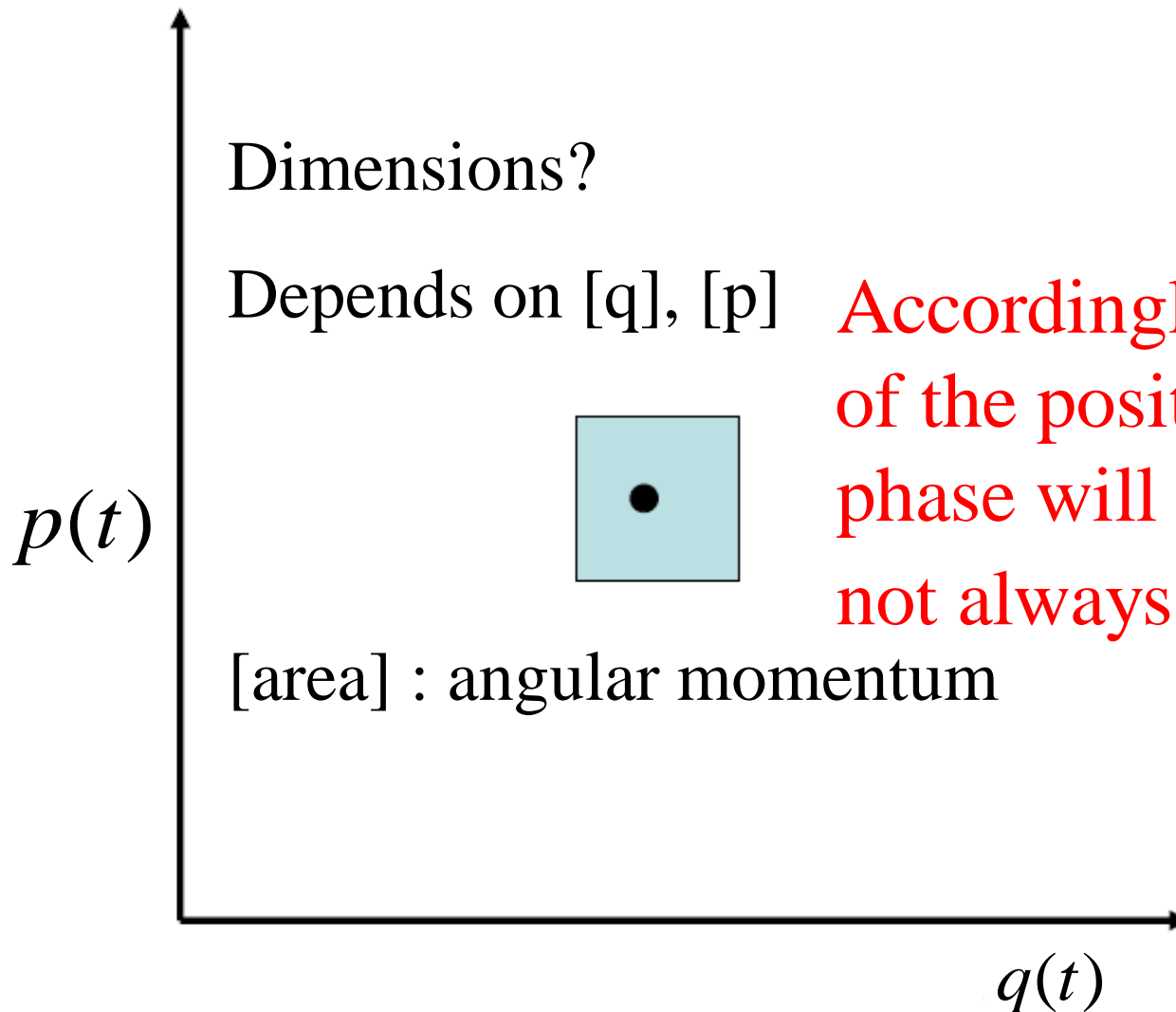

$\dot{x}(t)$

$x(t)$

The position-velocity phase space
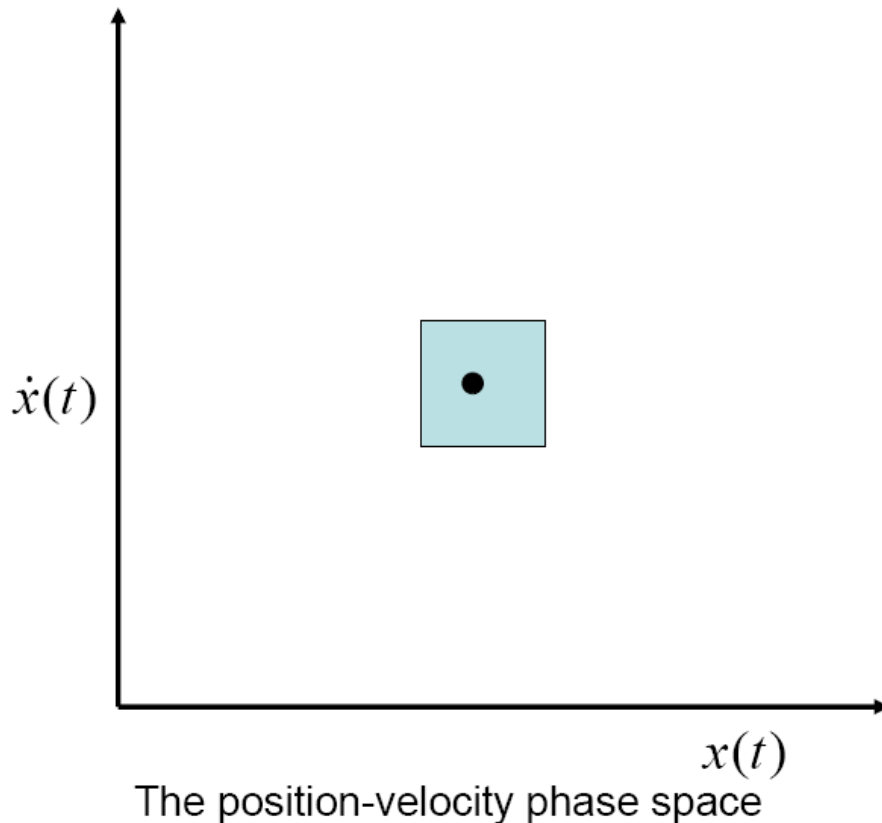
The physical dimensions of a piece of area in this space will **_not_** be L^2

Instead, the dimension of area in such a space would be

$$L^2 T^{-1}$$

position-momentum phase space

Dimensions?

Depends on [q], [p]

Accordingly, dimensions of the position-velocity phase will also not always be $L^2 T^{-1}$

$p(t)$

[area] : angular momentum

$q(t)$

The mechanical state of a system
is described by
a point in phase space.
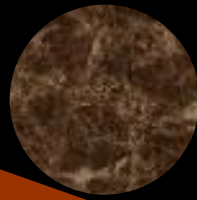
$Coordinates : (q, \dot{q}) \ or \ (q, p)$

$Evolution : (\dot{q}, \ddot{q}) \ or \ (\dot{q}, \dot{p})$
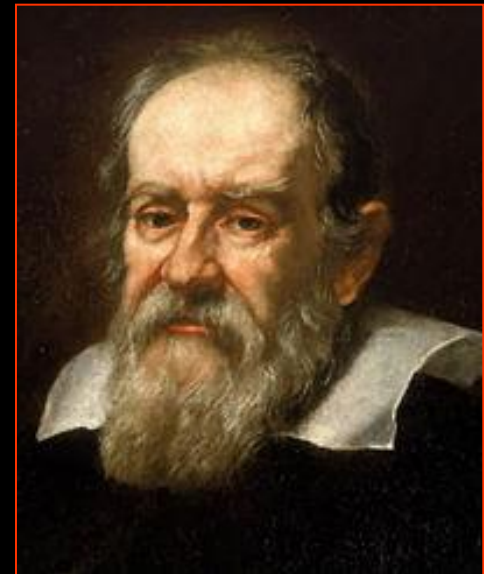
What is meant by 'Equation of Motion' ?

- Rigorous mathematical relationship between
      position, velocity and acceleration.

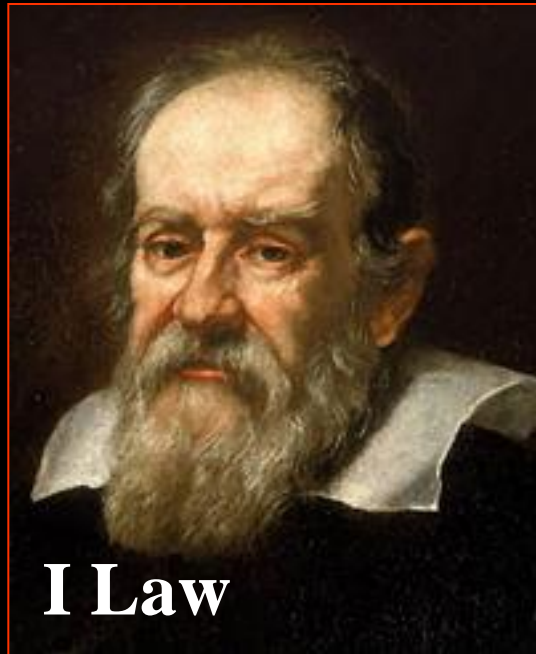**Galileo's experiments that led him to the law of inertia.**

Galileo Galilee
1564 - 1642

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What is 'equilibrium'?
What causes departure from 'equilibrium'?

**Galileo Galilei
1564 - 1642**

**Isaac Newton
(1642-1727)**
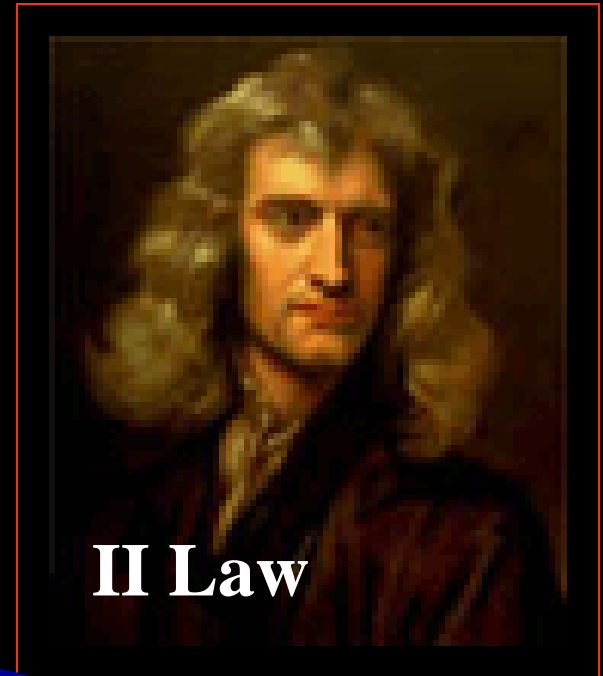
Causality
&
Determinism

I Law

II Law

$$\vec{F} = m\vec{a}$$ *Effect* is proportional to the C*ause*.

Linear Response.   Principle of causality.

PCD-10

$$\vec{F} = m\vec{a}$$ *Effect* is proportional to the C*ause*.

Linear Response.   Principle of causality.

Now, we already need calculus!

From Carl Sagan's 'Cosmos':

"Like Kepler, he (Newton) was not immune to the superstitions of his day… in 1663, at his age of twenty, he purchased a book on astrology, …. he read it until he came to an illustration which he could not understand, because he was ignorant of trigonometry. So he purchased a book on trigonometry but soon found himself unable to follow the geometrical arguments, So he found a copy of Euclid's *Elements of Geometry*, and began to read.

Two years later he invented the differential calculus."

$f(x)$
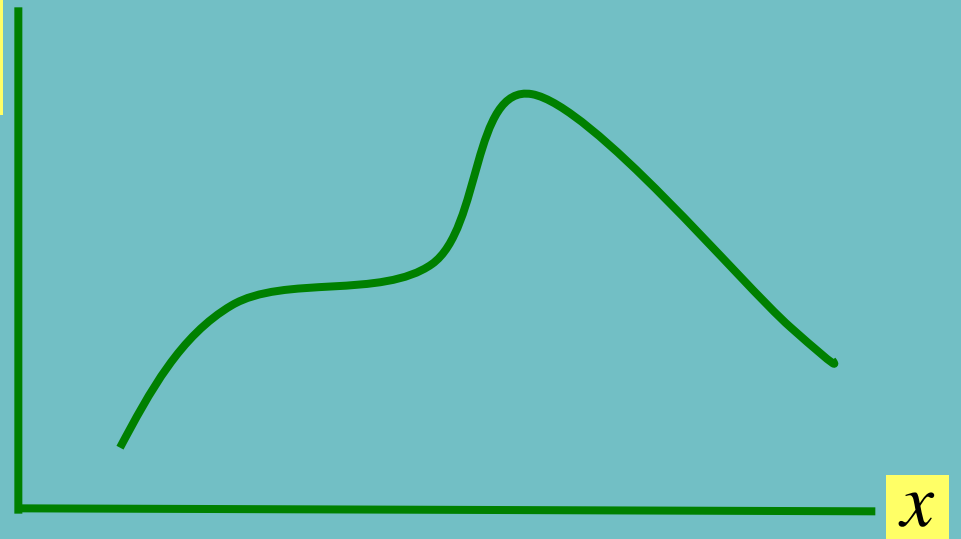
Differential of a function.
Derivative of a function
Slope of the curve.

x

It is a quantitative measure of how sensitively the function $f(x)$ responds to changes in the independent variable x .

The sensitivity may change from point to point and hence the derivative of a function must be determined at each value of x in the domain of x.

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$$\left[\frac{df}{dx}\right]_{x_0} = \lim_{\delta x \to 0}\left[\frac{\delta f}{\delta x}\right]_{x_0} = \lim_{\delta x \to 0}\frac{f(x_0 + \frac{\delta x}{2}) - f(x_0 - \frac{\delta x}{2})}{\delta x}$$

$f(x)$

$f(x_0)$

δf

δx

$$\delta f = \left[\frac{df}{dx}\right]_{x_0}\delta x$$

x

x_0
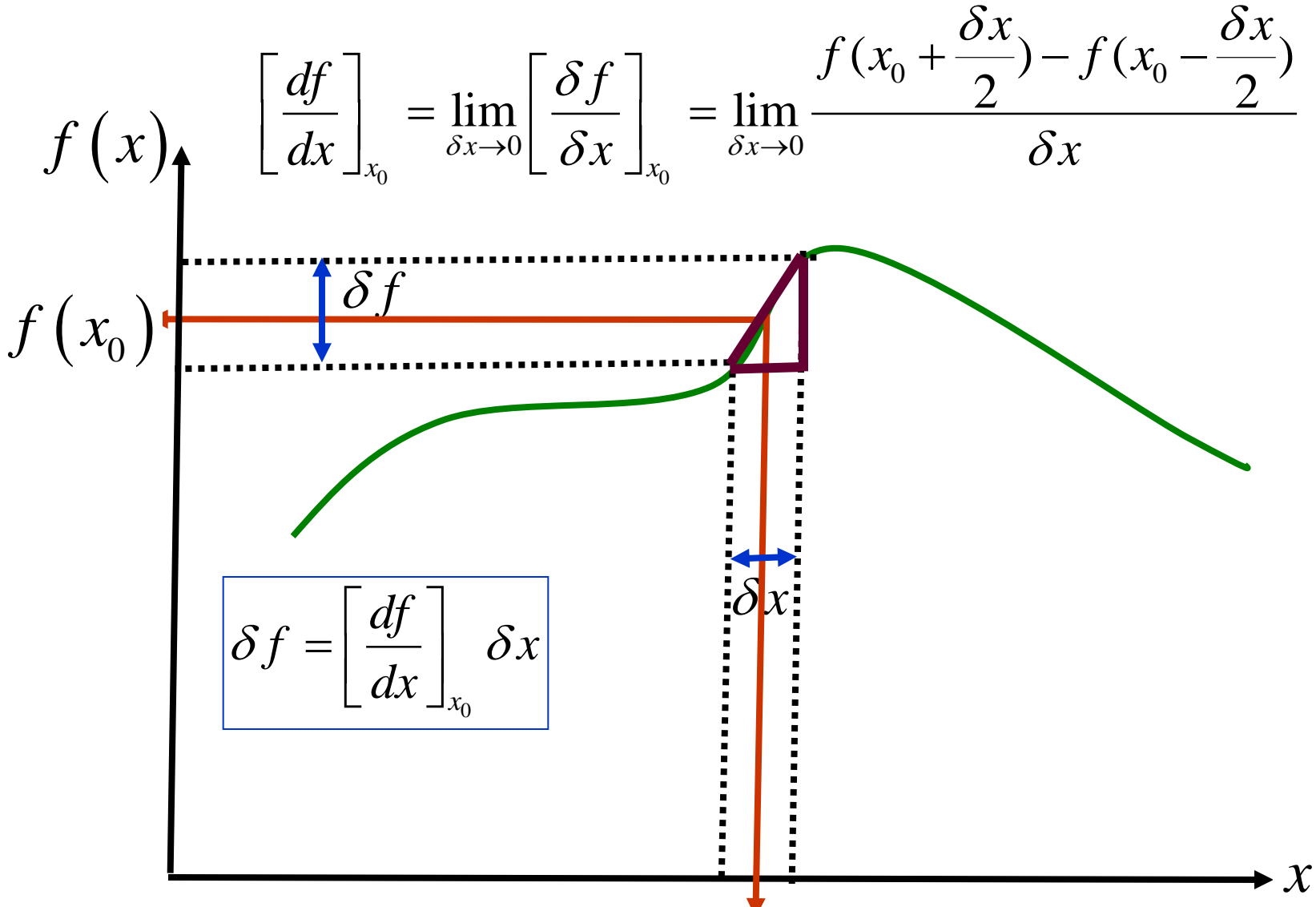
Tangent to the curve.
Dimensions of the derivative of the function: [f][x]$^{-1}$

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Functions of many variables: eg. height above a flat horizontal surface of a handkerchief that is stretched out in a warped surface, ……

h: dependent variable

x & y: independent variables

One must then define 'Partial derivatives' of a function of more than one variable.
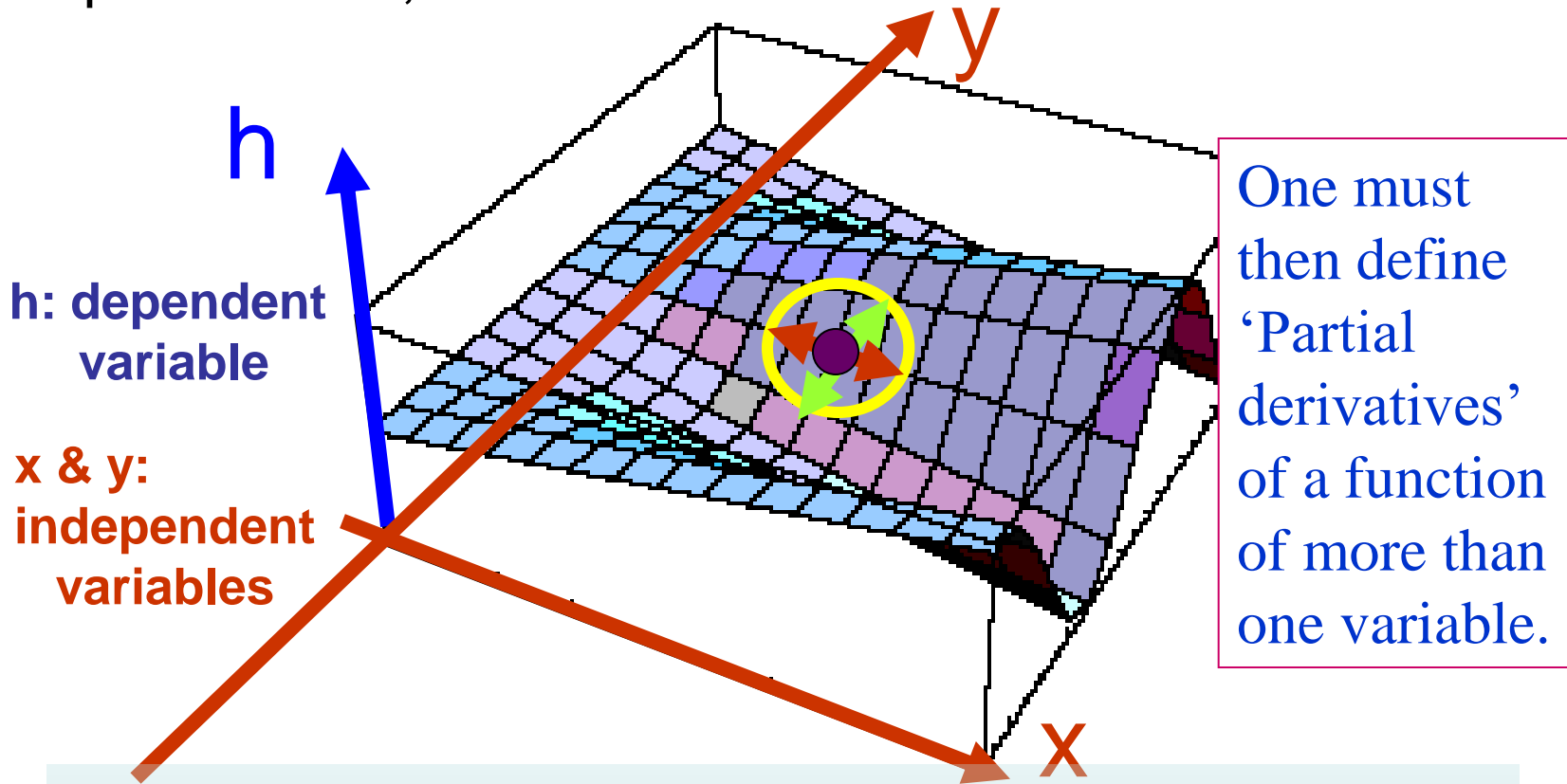
Or, the temperature on a flat surface that has various heat sources spread under it – such as a tiny hot filament here, a hotter there, a tiny ice cube here, a tiny beaker of liquid helium there, etc.

$$\left[\frac{\partial h}{\partial x}\right]_{(x_0,y_0)} = \lim_{\delta x \to 0} \frac{h(x_0 + \frac{\delta x}{2}, y_0) - h(x_0 - \frac{\delta x}{2}, y_0)}{\delta x} = \lim_{\delta x \to 0} \left[\frac{\delta h}{\delta x}\right]_{y_0}$$
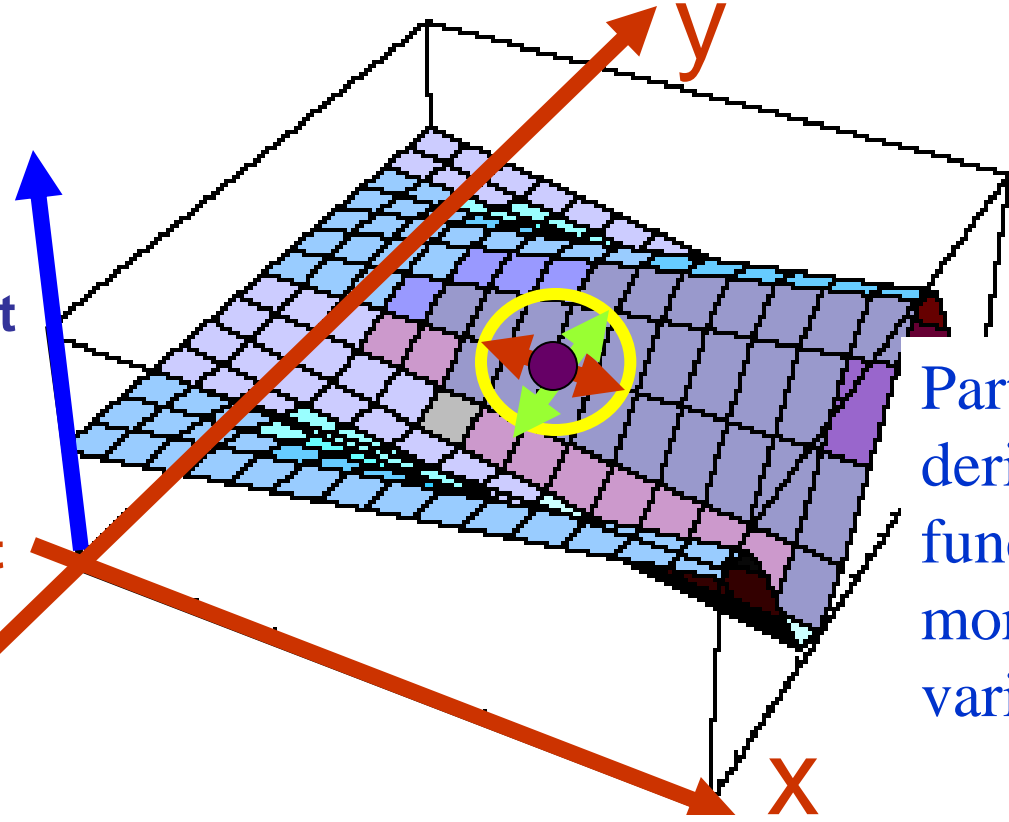
h

h: dependent variable

x & y: independent variables

Partial derivatives of a function of more than one variable.

y

x

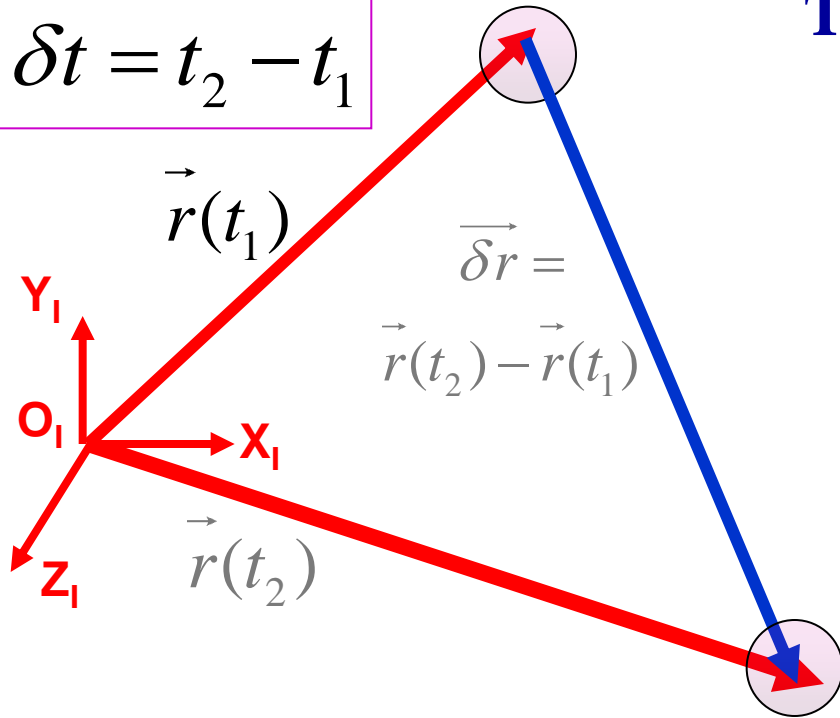$$\left[\frac{\partial h}{\partial y}\right]_{(x_0,y_0)} = \lim_{\delta y \to 0} \frac{h(x_0, y_0 + \frac{\delta y}{2}) - h(x_0, y_0 - \frac{\delta y}{2})}{\delta y} = \lim_{\delta y \to 0} \left[\frac{\delta h}{\delta y}\right]_{x_0}$$

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$$\delta t = t_2 - t_1$$

$$\vec{r}(t_1)$$

Y_I

O_I → **X_I**

Z_I

$$\vec{r}(t_2)$$

$$\overrightarrow{\delta r} = \vec{r}(t_2) - \vec{r}(t_1)$$

$$\vec{v} = \lim_{\delta t \to 0} \frac{\overrightarrow{\delta r}}{\delta t} \quad \longleftarrow \text{velocity}$$

$$\vec{v} = \lim_{\delta t \to 0} \frac{\vec{r}(t_2) - \vec{r}(t_1)}{\delta t}$$

'Equation of Motion'
Rigorous relationship between
      position,
      velocity
and   acceleration.

$$\vec{a} = \lim_{\delta t \to 0} \frac{\overrightarrow{\delta v}}{\delta t} \quad \longleftarrow \text{acceleration}$$

$$\vec{a} = \lim_{\delta t \to 0} \frac{\vec{v}(t_2) - \vec{v}(t_1)}{\delta t}$$

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I Law: Law of Inertia ----- COUNTER-INTUITIVE

What is 'equilibrium'?

Relative to whom?
- frame of reference

Equilibrium means
'state of rest',
or of uniform
motion along a
straight line.

Equilibrium sustains
itself, needs no cause;
determined entirely by
initial conditions.

$$\vec{F} = m\vec{a}$$

Effect $\vec{a}$ is proportional

to the Cause $\vec{F}$.

Proportionality: Mass/Inertia.

Linear Response.

Principle of causality/determinism.

Galileo; Newton

Force: Physical agency that changes the state of equilibrium of the object on which it acts.

$$m \frac{d^2 \vec{r}}{dt^2} = m \frac{d\vec{v}}{dt} = m\vec{a} = \vec{F} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}.$$

$$t \longrightarrow -t \qquad \frac{d}{dt} \rightarrow \left(-\frac{d}{dt} \right) \qquad \dot{\vec{r}} \rightarrow -\vec{v} \text{ and } \dot{\vec{v}} \rightarrow \left(-\frac{\vec{F}}{m} \right)$$

Direction of velocity and acceleration both reverse.

System's trajectory would be only reversed along essentially the same path.

Newton's laws are therefore symmetric under time-reversal.

Newton's laws: 'fundamental' principles
– *laws of nature*.

Did we derive Newton's laws from anything?

We got the first law from Galileo's brilliant experiments.

We got the second law from Newton's explanation of departure from equilibrium in terms of the linear-response cause-effect relationship.

No violation of these predictions was ever found.

- especially elevated status: 'laws of nature';
- universal in character since no mechanical phenomenon seemed to be in its range of applicability.

We have introduced the first two laws
of Newton as fundamental principles.

Newton's III law makes a qualitative and quantitative
statement about each pair of interacting objects, which
exert a mechanical force on each other.

$$\vec{F}_{12} = -\vec{F}_{21}$$

In Newtonian scheme of
mechanics, this is introduced
as a 'fundamental' principle
–i.e., as *a law of nature*.

Newton's III law :

'Action and Reaction are Equal and Opposite'

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}}$$

$$\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt}$$

$$\frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = 0$$

Newton's III Law as statement of conservation of linear momentum

We have obtained a conservation principle from 'law of nature'

Are the conservation principles consequences of the laws of nature? Or, are the laws of nature the consequences of the symmetry principles that govern them?

Until Einstein's special theory of relativity,
it was believed that
conservation principles are the result of the laws of nature.

Since Einstein's work, however, physicists began to analyze the conservation principles as consequences of certain underlying symmetry considerations,
enabling the laws of nature to be revealed from this analysis.

Instead of introducing Newton's III law as a *fundamental principle,*

we shall now *deduce* it from symmetry / invariance.

This approach places SYMMETRY *ahead of* LAWS OF NATURE.
It is this approach that is of greatest value to contemporary physics. This approach has its origins in the works of Albert Einstein, Emmily Noether and Eugene Wigner.

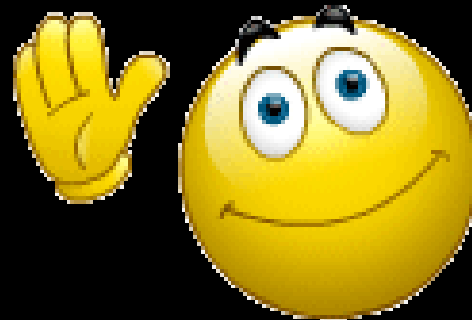**(1879 – 1955)**   **(1882 – 1935)**   **(1902 – 1995)**

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NEXT CLASS:

STiCM Lecture 03: Unit 1 Equations of Motion (ii)

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Newton's III law :

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enabling the laws of nature to be revealed from this analysis.

Before we proceed,
we remind ourselves of another illustration of the
connection between symmetry and conservation law.

Examine the ANGULAR MOMENTUM $\vec{l} = \vec{r} \times \vec{p}$
of a system subjected to a central force.

$$\vec{\tau} = \frac{d\vec{l}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \vec{r} \times \vec{F},$$

$$\vec{F} = \left|\vec{F}\right|\hat{e}_r \qquad \text{SYMMETRY}$$

$$\Rightarrow \vec{\tau} = \vec{0}$$

since $\dfrac{d\vec{r}}{dt} \times \vec{p} = \vec{0}$ and $\dfrac{d\vec{p}}{dt} = \vec{F}.$

$\vec{l}$: constant $\qquad$ CONSERVED QUANTITY

The connection between 'symmetry' and 'conservation law' is so intimate, that we can actually *derive* Newton's III law using 'symmetry'.

Begin with a symmetry principle:

translational invariance in homogenous space.

Consider a system of N particles in a medium that is homogenous.

A displacement of the entire N-particle system through $\overrightarrow{\delta S}$ in this medium would result in a new configuration that would find itself in an environment that is completely indistinguishable from the previous one.

This invariance of the environment of the entire N particle system following a translational displacement is a result of translational symmetry in homogenous space.

FOUR essential considerations:

1. Each particle of the system is under the influence *only* of all the remaining particles;
the system is isolated: no external forces act on any of its particles.

2. The entire N-particle system is deemed to have undergone simultaneous identical displacement;
all inter-particle separations and relative orientations remain invariant.

3. Entire medium: essentially homogeneous;
   - spanning the entire system both
   before and after the displacement.

4. The displacement the entire system under consideration is deemed to take place at a certain instant of time.

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The implication of these FOUR considerations:

"Displacement considered is only a *virtual displacement*."

- only a mental thought process;

- real physical displacements would require a certain time interval over which the displacements would occur

- virtual displacements can be thought of to occur *at an instant of time* and subject to the specific four features mentioned.

Displacement being virtual,

it is redundant to ask

"what agency has caused it"

Now, the internal forces do <u>no work</u> in this virtual displacement.

Therefore 'work done' by the internal forces in the 'virtual displacement' must be zero.

This work done *(rather <u>not done</u>)*
is called 'virtual work'.

$$0 = \delta W = \left\{ \sum_{k=1}^{N} \vec{F}_k \right\} \bullet \vec{\delta s} = \left\{ \sum_{k=1}^{N} \sum_{i=1, i \neq k}^{N} \vec{F}_{ik} \right\} \bullet \vec{\delta s},$$

where $\vec{F}_{ik}$: force on k^{th} particle by the ith,

and

$\vec{F}_k$ force on the kth particle due to the

remaining N -1 particles.

$$0 = \delta W = \left\{ \sum_{k=1}^{N} \vec{F}_k \right\} \bullet \overrightarrow{\delta s} = \left\{ \sum_{k=1}^{N} \sum_{i=1, i \neq k}^{N} \vec{F}_{ik} \right\} \bullet \overrightarrow{\delta s}$$

The mathematical techniques: Jean le Rond d'Alembert (1717 – 1783)

Under what conditions can the above relation hold for an ARBITRARY displacement $\overrightarrow{\delta s}$?

$$\vec{0} = \left\{ \sum_{k=1}^{N} \vec{F}_k \right\} = \sum_{k=1}^{N} \frac{d\vec{p}_k}{dt} = \frac{d}{dt} \sum_{k=1}^{N} \vec{p}_k = \frac{d\vec{P}}{dt}$$

Newton's I,II laws used; not the III.

The relation
$$\vec{0} = \left\{ \sum_{k=1}^{N} \vec{F}_k \right\} = \sum_{k=1}^{N} \frac{d\vec{p}_k}{dt} = \frac{d}{dt} \sum_{k=1}^{N} \vec{p}_k = \frac{d\vec{P}}{dt}$$

is obtained from the properties of translational symmetry in homogenous space.

Amazing!

For just two particles:

$$\frac{d\vec{P}}{dt} = \vec{0},$$

- since it suggests a path to

discover the laws of physics

i.e., $\dfrac{d\vec{p}_2}{dt} = -\dfrac{d\vec{p}_1}{dt},$

by exploiting the connection

between symmetry and

which gives $\vec{F}_{12} = -\vec{F}_{21},$
the *III law of Newton.*

conservation laws!

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Newton's III law need not be introduced as a *fundamental principle/law;*

we *deduced* it from symmetry / invariance.

SYMMETRY placed *ahead of* LAWS OF NATURE.

Albert Einstein,   Emmily Noether   and   Eugene Wigner.

(1879 – 1955)   **(1882 – 1935)**   **(1902 – 1995)**

Emilly Noether: Symmetry ⟷ Conservation Laws

Eugene Wigner's profound impact on physics:

symmetry considerations using `group theory' resulted

in a change in the very perception of just what is most

fundamental.

`symmetry' : the most fundamental entity whose form

would govern the physical laws.

The connection between SYMMETRY and CONSERVATION PRINCIPLES brought out in the previous example,
becomes even more transparent in an *alternative scheme* of MECHANICS.

While Newtonian scheme rests on the principle of causality (effect is linearly proportional to the cause), this alternative principle does *not* invoke the notion of 'force' as the 'cause' that must be invoked to explain a system's evolution.

This alternative principle also begins by the fact that the mechanical system is characterized by its position and velocity (or equivalently by position and momentum),

…… but with a very slight difference!

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The mechanical system is determined by a **well-defined function** of position and velocity/momentum.

The functions that are employed are the Lagrangian $L(q, \dot{q})$ and the Hamiltonian $H(q, p)$

The primary principle on which this

alternative formulation rests is known as the

Principle of Variation.

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We shall now introduce the
'Principle of Variation',

often referred to as the
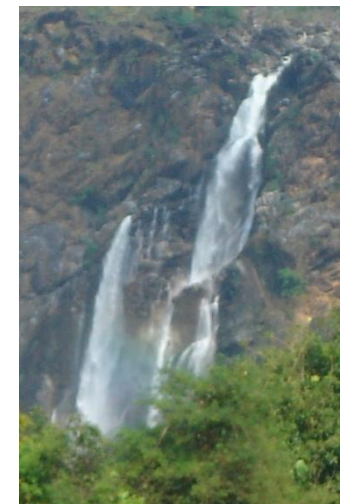'Principle of Least (extremum) Action'.

The principle of least (extremum) action: *in its various incarnations applies to all of physics*.

- explains why things happen the way they do!

-Explains trajectories of mechanical systems subject to certain initial conditions.

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The principle of least action has for its precursor what is known as Fermat's principle - explains why light takes the path it does when it meets a boundary of a medium.

Common knowledge: when a ray of light meets the edge of a medium, it usually does not travel along the direction of incidence - gets reflected and refracted.

Fermat's principle explains this by stating that light travels from one point to another along a path over which it would need the '*least*' time.

Pierre de Fermat (1601(?)-1665) : French lawyer who pursued mathematics as an active hobby.

Best known for what has come to be known as Fermat's last theorem, namely that the equation $x^n+y^n=z^n$ has no non-zero integer solutions for x,y and z for any value of n>2.

"To divide a cube into two other cubes, a fourth power or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it."
                    -- Pierre de Fermat

It took about 350 years for this theorem to be proved (by Andrew J. Wiles, in 1993).

Actually, the time taken by light is not necessarily a minimum.

More correctly, the principle that we are talking about is stated in terms of an 'extremum',

and even more correctly as

'The actual ray path between two points is the one for which the optical path length is stationary with respect to variations of the path'.

………………. Usually it is a minimum:
Light travels along a path that takes the least time.

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STiCM Lecture 04: Unit 1 Equations of Motion (iii)

We shall now introduce the
'Principle of Variation',

often referred to as the
'Principle of Least (extremum) Action'.

Hamilton's principle of least action thus has an interesting development, beginning with Fermat's principle about how light travels between two points, and rich contributions made by Pierre Louis Maupertuis (1698 -- 1759), Leonhard Paul Euler (1707 - 1783), and Lagrange himself.

Hamilton's principle

'principle of least *(rather, extremum)* action'

Hamilton's principle

'principle of least *(rather, extremum)* action'

William Rowan Hamilton
(1805 - 1865)

Mechanical state of a system 'evolves' (along a 'world line') in such a way that
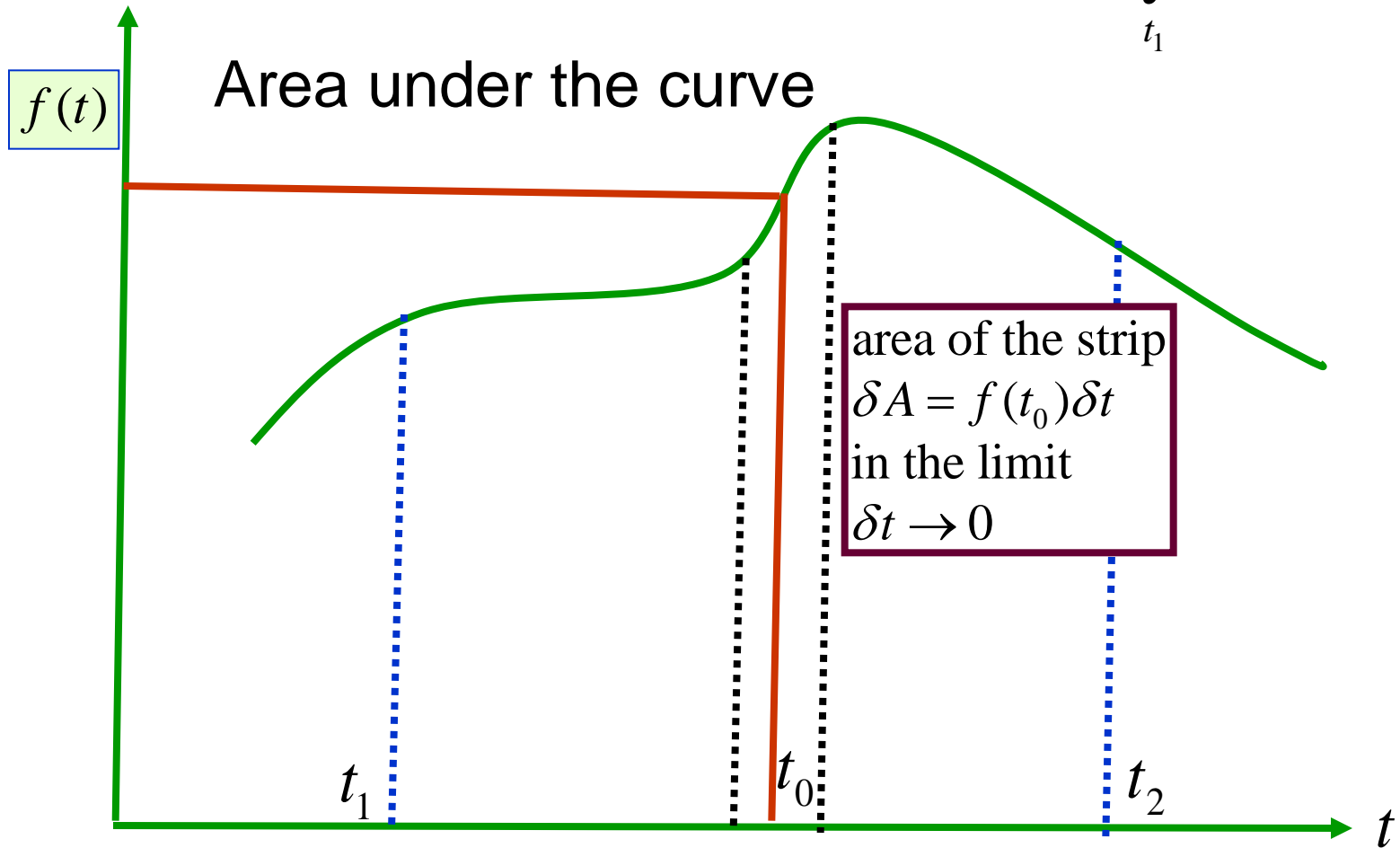
$$'action', \quad S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \quad \text{is an extremum.}$$

...and now, we need 'action',
— 'integral' of the 'Lagrangian'!

Integral of a function of time.

$$Definite\ \text{integral of the function} = \int_{t_1}^{t_2} f(t)dt$$

$f(t)$

Area under the curve

area of the strip
$\delta A = f(t_0)\delta t$
in the limit
$\delta t \to 0$

t_1 t_0 t_2 t

Dimensions of this 'area': $\left[f(t) \right] \times T$

Path Integral. $\quad 'action', \ S = \int\limits_{t_1}^{t_2} L(q,\dot{q},t)dt$

Dependence of L on time is not explicit. It is implicit through dependence on position and velocity which depend on t.

$$'action', \ S = \int\limits_{t_1}^{t_2} L(q(t),\dot{q}(t),t)dt$$

The system evolution cannot be shown on a two-dimensional surface.

The system then evolves along a path in the 'phase space'.

The additive property of 'action' as area under the L vs. time curve remains applicable.

Thus, the dimensions of 'action' are equal to dimensions of the Lagrangian multiplied by T.

We shall soon discover what L is!

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$\dot{q}(t)$

$q(t_1), \dot{q}(t_1)$

$q(t_2), \dot{q}(t_2)$
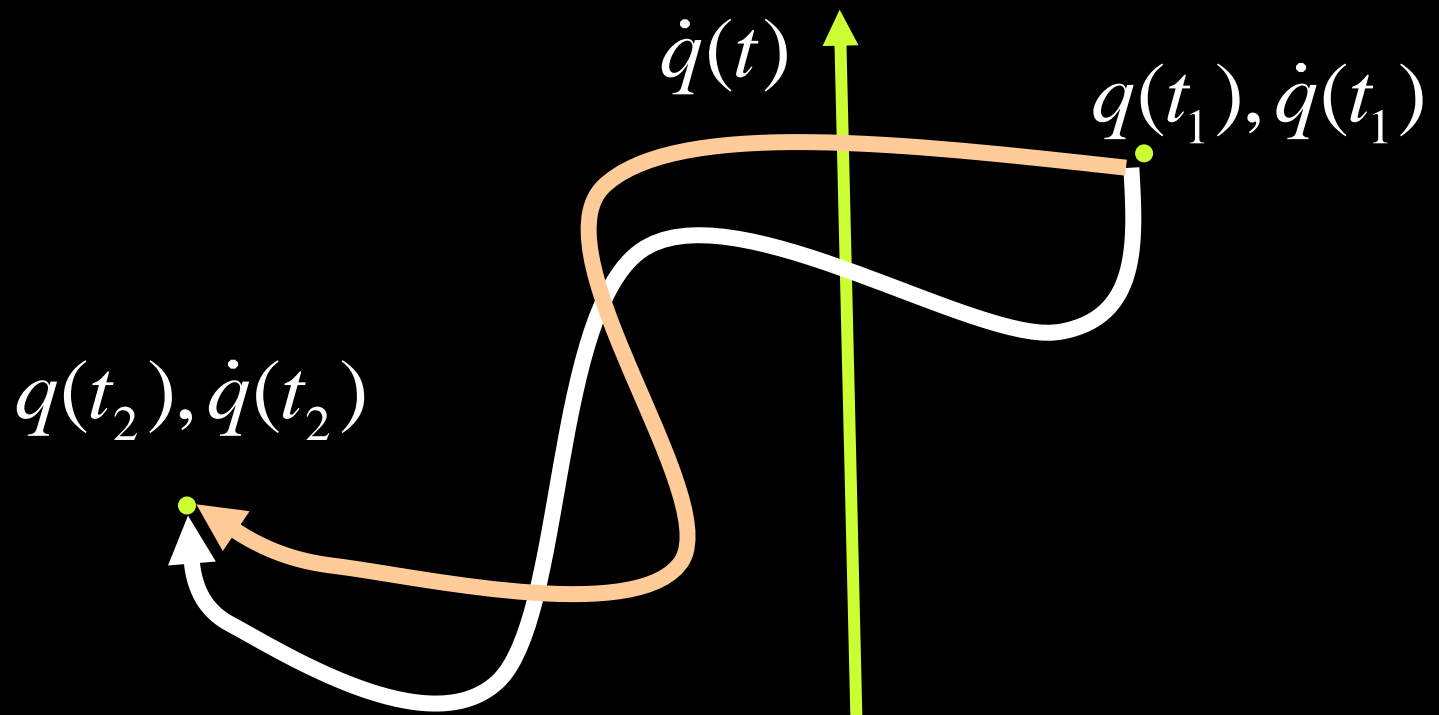
Consider alternative paths along which the mechanical state of the system may evolve in the phase space :

q changed by δq,

and $\dot{q}$ *changed by* $\delta \dot{q}$

$q(t)$

$$\boxed{\begin{array}{l} \textit{Alternative paths}: \\ q \ \textit{changed by } \ \delta q, \\ \textit{and } \ \dot{q} \ \textit{changed by } \ \delta\dot{q} \end{array}}$$

The mechanical system <u>evolves</u> in such a way that

$$'action', \ S = \int_{t_1}^{t_2} L(q,\dot{q},t)dt \ \text{ is an extremum}$$

Note! 'Force', 'Cause-Effect Relationship' is NOT invoked!

S would be an extremum
when the variation in S is zero;
i.e. $\delta S = 0$

$$\boxed{\delta S = \int_{t_1}^{t_2} L(q+\delta q, \dot{q}+\delta\dot{q}, t)dt - \int_{t_1}^{t_2} L(q,\dot{q},t)dt = 0}$$
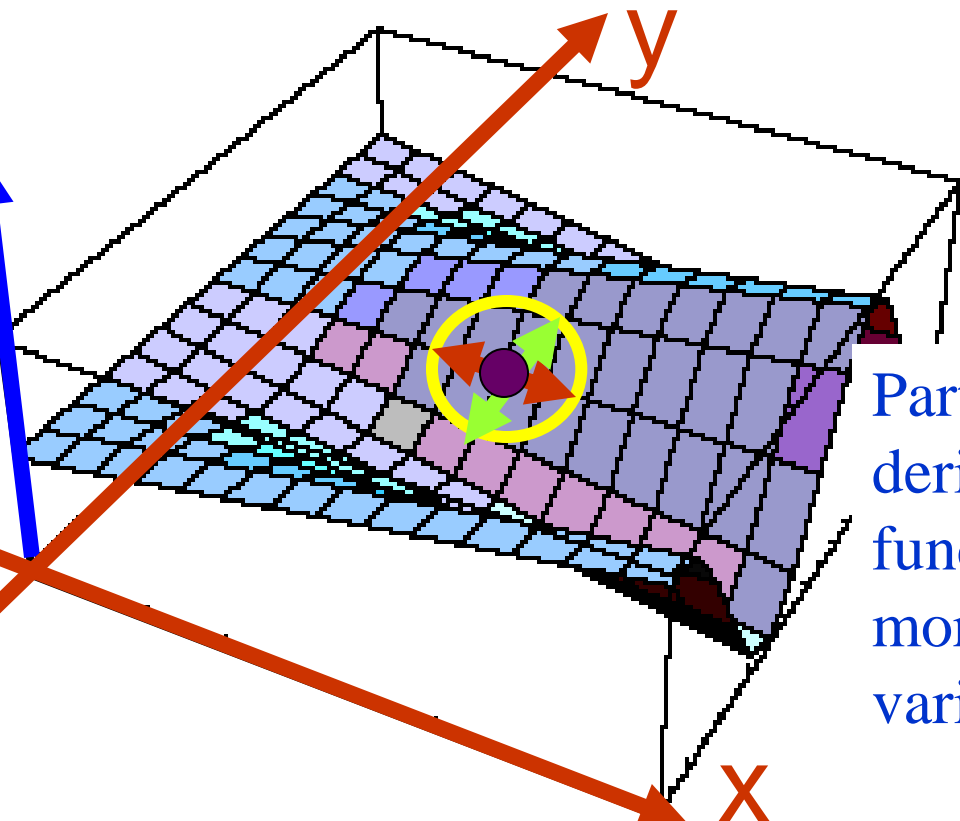
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$$\delta f = \frac{df}{dx} \delta x$$

$$\delta h(x, y) = \frac{\partial h}{\partial x} \delta x + \frac{\partial h}{\partial y} \delta y$$

$$\left[\frac{\partial h}{\partial x}\right]_{(x_0,y_0)} = \lim_{\delta x \to 0} \frac{h(x_0 + \frac{\delta x}{2}, y_0) - h(x_0 - \frac{\delta x}{2}, y_0)}{\delta x} = \lim_{\delta x \to 0}\left[\frac{\delta h}{\delta x}\right]_{y_0}$$

h

h: dependent variable

x & y: independent variables

Partial derivatives of a function of more than one variable.

y

x

$$\left[\frac{\partial h}{\partial y}\right]_{(x_0,y_0)} = \lim_{\delta y \to 0} \frac{h(x_0, y_0 + \frac{\delta y}{2}) - h(x_0, y_0 - \frac{\delta y}{2})}{\delta y} = \lim_{\delta y \to 0}\left[\frac{\delta h}{\delta y}\right]_{x_0}$$
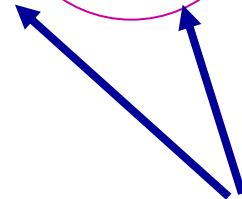
$$\delta S = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t)\, dt - \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt = 0$$

i.e., $\quad 0 = \delta S = \int_{t_1}^{t_2} \delta L(q, \dot{q}, t)\, dt$

$$0 = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right\} dt$$

$$0 = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \left(\delta q \right) \right\} dt$$

We need: Integration of product of two functions

differential and integral of a product of two functions.

$$\frac{d}{dx}\{f(x)g(x)\} = \left\{\frac{df}{dx}\right\}g(x) + f(x)\left\{\frac{dg}{dx}\right\}$$

$$\therefore$$

$$f(x)\left\{\frac{dg}{dx}\right\} = \frac{d}{dx}\{f(x)g(x)\} - \left\{\frac{df}{dx}\right\}g(x)$$

$$f(x)\left\{\frac{dg}{dx}\right\} = \frac{d}{dx}\{f(x)g(x)\} - \left\{\frac{df}{dx}\right\}g(x)$$

Integrating both sides:

$$\int f(x)\left\{\frac{dg}{dx}\right\}dx = \int \frac{d}{dx}\{f(x)g(x)\}dx - \int \left\{\frac{df}{dx}\right\}g(x)dx$$

$$\int_{x_1}^{x_2} f(x)\left\{\frac{dg}{dx}\right\}dx = f(x)g(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2}\left\{\frac{df}{dx}\right\}g(x)dx$$

$$\int_{x_1}^{x_2} f(x)\left\{\frac{dg}{dx}\right\}dx = f(x_2)g(x_2) - f(x_1)g(x_1) - \int_{x_1}^{x_2}\left\{\frac{df}{dx}\right\}g(x)dx$$

PCD-10

$$0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \boxed{\delta \dot{q}} \right\} dt = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} (\delta q) \right\} dt$$

$$i.e. \quad 0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt + \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial \dot{q}} \frac{d(\delta q)}{dt} \right\} dt$$

Integration of product of two functions

$$\int_{x_1}^{x_2} f(x) \left\{ \frac{dg}{dx} \right\} dx = f(x) g(x) \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \left\{ \frac{df}{dx} \right\} g(x) dx$$

$$0 = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt +$$

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$$0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \boxed{\delta \dot{q}} \right\} dt = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \boxed{\frac{d}{dt} (\delta q)} \right\} dt$$
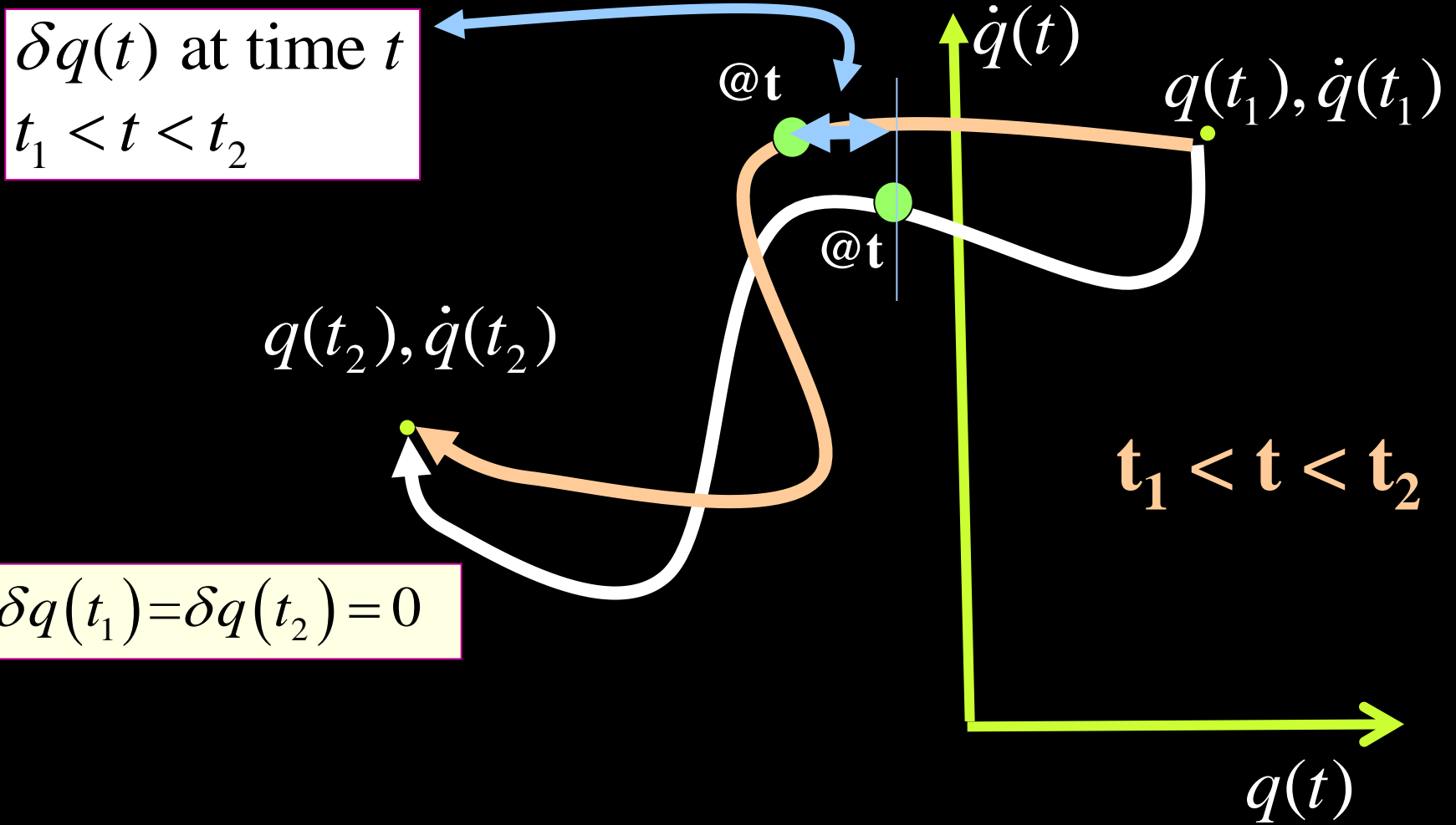
$$i.e. \quad 0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt + \int_{t_1}^{t_2} \left\{ \boxed{\frac{\partial L}{\partial \dot{q}}} \boxed{\frac{d(\delta q)}{dt}} \right\} dt$$

Integration of product of two functions

$$\int_{x_1}^{x_2} f(x) \left\{ \frac{dg}{dx} \right\} dx = f(x) g(x) \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \left\{ \frac{df}{dx} \right\} g(x) dx$$

$$0 = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt + \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right\} dt$$

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$\delta q(t)$ at time t
$t_1 < t < t_2$

@t

$\dot{q}(t)$

$q(t_1), \dot{q}(t_1)$

@t

$q(t_2), \dot{q}(t_2)$

$\mathbf{t_1 < t < t_2}$

$\delta q(t_1) = \delta q(t_2) = 0$

$q(t)$

$$0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt + \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right\} dt$$

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Select / Special Topics in Classical Mechanics

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STiCM Lecture 05: Unit 1 Equations of Motion (iv)

Mechanical state of a system 'evolves'
(along a 'world line') in such a way that

$$'action', \quad S = \int_{t_1}^{t_2} L(q,\dot{q},t)dt \quad \text{is an extremum.}$$

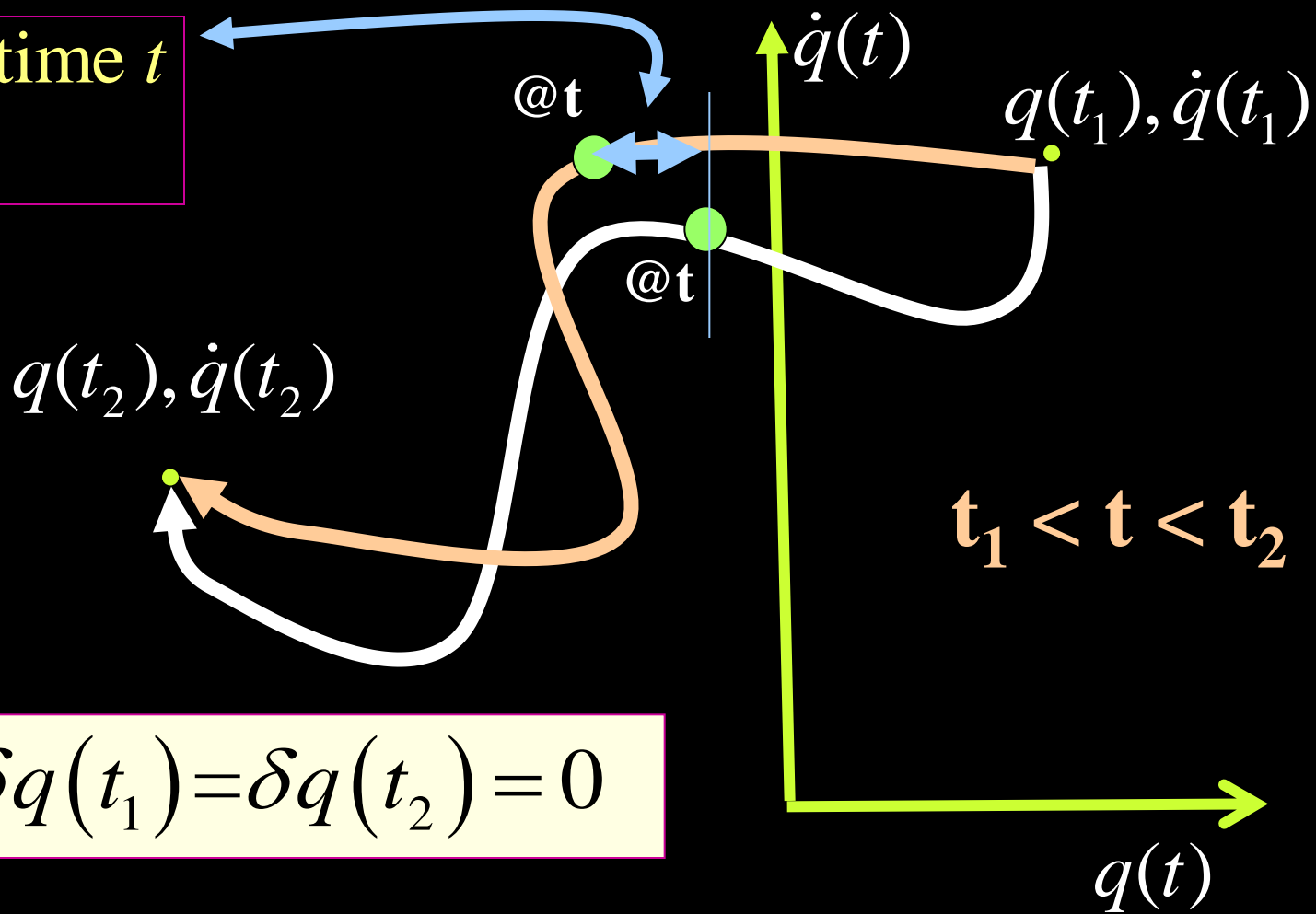$$\delta S = \int_{t_1}^{t_2} L(q+\delta q, \dot{q}+\delta\dot{q}, t)dt - \int_{t_1}^{t_2} L(q,\dot{q},t)dt = 0$$

$$\text{i.e.,} \quad 0 = \delta S = \int_{t_1}^{t_2} \delta L(q,\dot{q},t)dt$$

$$0 = \delta S = \int_{t_1}^{t_2}\left\{\frac{\partial L}{\partial q}\delta q\right\}dt + \left[\frac{\partial L}{\partial \dot{q}}\,\delta q\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q\right\}dt$$

$\delta q(t)$ at time t
$t_1 < t < t_2$

@t

$\dot{q}(t)$

$q(t_1), \dot{q}(t_1)$

$q(t_2), \dot{q}(t_2)$

@t

$\mathbf{t_1 < t < t_2}$

$q(t)$

$\delta q(t_1) = \delta q(t_2) = 0$

$$0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt + \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right\} dt$$

$$0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt + \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right\} dt$$

$$0 = \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} \delta q \right\} dt - \int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \delta q \right\} dt$$

i.e. $\quad 0 = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \right\} \delta q \, dt$

Arbitrary variation

between the end points.

Hence, $\dfrac{\partial L}{\partial q} - \dfrac{d}{dt} \left(\dfrac{\partial L}{\partial \dot{q}} \right) = 0$

Lagrange's Equation

We have not, as yet,

provided a recipe to

construct the

Lagrangian!

$L = L(q, \dot{q})$ is all the we know about it as yet!

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \qquad \textit{Lagrange's Equation} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

Homogeneity & Isotropy of space

L can only be quadratic function of the velocity.

$$L(q, \dot{q}, t) = f_1(\dot{q}^2) + f_2(q)$$

$$L(q, \dot{q}, t) = \frac{m}{2}\dot{q}^2 - V(q)$$

$$= T - V$$

Lagrange's Equation $\quad \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) = \dfrac{\partial L}{\partial q}$

$$L(q, \dot{q}, t) = \frac{m}{2}\dot{q}^2 - V(q)$$

$$= T - V$$

$\dfrac{\partial L}{\partial q} = -\dfrac{\partial V}{\partial q} = F$, the force

$\dfrac{\partial L}{\partial \dot{q}} = m\dot{q} = p$, the momentum

i.e., $\dfrac{dp}{dt} = F$: in 3D: $\dfrac{d\vec{P}}{dt} = \vec{F}$

Newton's II Law

$$L(q, \dot{q}, t) = \frac{m}{2} \dot{q}^2 - V(q) \qquad p = \frac{\partial L}{\partial \dot{q}},$$

$$= T - V \qquad \textit{generalized momentum}$$

$$\frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q} = F, \text{ the force}$$

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} = p, \text{ the momentum}$$

Interpretation of L as T-V gives equivalent correspondence with Newtonian formulation.

$$L = L(q, \dot{q}, t)$$

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \left(\frac{\partial L}{\partial q}\right)\dot{q} + \left(\frac{\partial L}{\partial \dot{q}}\right)\ddot{q} + \frac{\partial L}{\partial t}$$

$$0 = \left(\frac{\partial L}{\partial q}\right) - \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\right)$$

$$\left(\frac{\partial L}{\partial q}\right) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}}\right)$$

$$\frac{dL}{dt} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial L}{\partial \dot{q}}\right\}\right)\dot{q} + \left(\frac{\partial L}{\partial \dot{q}}\right)\ddot{q} + \frac{\partial L}{\partial t}$$

$$= \frac{d}{dt}\left(\left\{\frac{\partial L}{\partial \dot{q}}\right\}\dot{q}\right) + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left(\left\{\frac{\partial L}{\partial \dot{q}}\right\}\dot{q} - L\right) = -\frac{\partial L}{\partial t}$$

$$\textit{What if } \frac{\partial L}{\partial t} = 0?$$

$$\left\{ \frac{\partial L}{\partial \dot{q}} \right\} \dot{q} \text{ - } L \text{ is CONSTANT} \quad \Longleftarrow \quad \frac{\partial L}{\partial t} = 0$$

Hamiltonian's Principal Function

$$H = \left[\left\{ \frac{\partial L}{\partial \dot{q}} \right\} \dot{q} \text{ - } L \right] = p\dot{q} \text{ - } L$$

$$H = m\mathrm{v}^2 - L$$

$$= m\mathrm{v}^2 - \left(\frac{1}{2} m\mathrm{v}^2 - V \right)$$

$$H = 2T \text{ - } L = 2T \text{ - } (T \text{ - } V) = T + V$$

$$\frac{\partial L}{\partial q} = -\frac{\partial V}{\partial q} = F, \text{ force}$$

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q} = p, \text{ momentum}$$

TOTAL ENERGY

When there are

N degrees *of freedom,*

$$H = \sum_{i=1}^{N} p_i \dot{q}_i - L$$

$$p = \frac{\partial L}{\partial \dot{q}}$$

q : *Generalized Coordinate*

$\dot{q}$: *Generalized Velocity*

p : *Generalized Momentum*

Symmetry ↔ Conservation Laws

(Noether)

$$\frac{\partial L}{\partial t} = 0$$

Lagrangian of a closed system does not depend explicitly on time.

$$\left[\dot{q} \frac{\partial L}{\partial \dot{q}} - L \right] \text{ is a CONSTANT}$$

Hamiltonian: "ENERGY"

Conservation of Energy is thus connected with the symmetry principle regarding invariance with respect to temporal translations.

Hamiltonian / Hamilton's Principal Function

In an inertial frame, Time: homogeneous;

Space is homogenous and isotropic

the condition for homogeneity of space : $\delta L(x, y, z) = 0$

$$i.e., \quad \delta L = \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial y} \delta y + \frac{\partial L}{\partial z} \delta z = 0$$

which implies $\boxed{\dfrac{\partial L}{\partial q} = 0}$ where $q = x, y, z$

since $\dfrac{\partial L}{\partial q} - \dfrac{d}{dt} \left(\dfrac{\partial L}{\partial \dot{q}} \right) = 0$, this means $i.e.$ $\boxed{\dfrac{\partial L}{\partial \dot{q}} = p}$ is conserved.

$i.e.$, is independent of time, is a constant of motion

$$\frac{\partial L}{\partial q} = 0$$

since $\dfrac{\partial L}{\partial q} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) = 0,$ this means *i.e.* $\dfrac{\partial L}{\partial \dot{q}} = p$ is conserved.

i.e., p is independent of time, it is a constant of motion

Law of conservation of momentum,
arises from the homogeneity of space.

Symmetry ⟷ Conservation Laws

Momentum that is canonically conjugate to a
cyclic coordinate is conserved.

Many degrees of freedom: $H = \sum_k \left[\dot{q}_k p_k - L(q_k, \dot{q}_k) \right]$

$$dH = \sum_k p_k \, d\dot{q}_k + \sum_k \dot{q}_k \, dp_k - \sum_k \frac{\partial L}{\partial q_k} \, dq_k - \sum_k \frac{\partial L}{\partial \dot{q}_k} \, d\dot{q}_k$$

$$dH = \sum_k \dot{q}_k \, dp_k - \sum_k \frac{\partial L}{\partial q_k} \, dq_k$$

$$= \sum_k \dot{q}_k \, dp_k - \sum_k \dot{p}_k \, dq_k$$

$$dH = \sum_k \dot{q}_k \; dp_k - \sum_k \dot{p}_k \; dq_k$$

$$But, \; H = H(p_k, q_k)$$

$$so \; dH \;\; = \;\; \sum_k \frac{\partial H}{\partial p_k} dp_k + \sum_k \frac{\partial H}{\partial q_k} dq_k$$

$$Hence \;\; \forall \; k: \;\; \frac{\partial H}{\partial p_k} = \dot{q}_k \;\; and \;\; \frac{\partial H}{\partial q_k} = -\dot{p}_k$$

Hamilton's equations of motion

Hamilton's Equations

$$H = \sum_k \left[q_k p_k - L(q_k, \dot{q}_k) \right]$$

$$H = H(p_k, q_k)$$

$$\forall \, k : \quad \frac{\partial H}{\partial p_k} = \dot{q}_k \quad and \quad \frac{\partial H}{\partial q_k} = -\dot{p}_k$$

Hamilton's Equations of Motion :

Describe how a mechanical state of a system

characterized by (*q,p*) 'evolves' with time.

On the extremality of the action integral

A Galindo and G García Alcaine

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Abstract. Some necessary and sufficient conditions for a critical point of the action integral to be locally or globally extremal are proved. Applications to systems with finite or infinite number of degrees of freedom are discussed.

Extremum – Maximum ?
Minimum ?
Saddle Point ?
Stationarity……

Locally stable? Globally stable?

Advanced concepts – left out for future studies…

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NEXT CLASS: STiCM Lecture 06: Unit 1 Equations of Motion (v)

STiCM

Select / Special Topics in Classical Mechanics

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STiCM Lecture 06: Unit 1 Equations of Motion (v)

Applications of Lagrange's/Hamilton's Equations

➡ Entire domain of Classical Mechanics

➡ Enables emergence of '*Conservation of Energy*'

     and   'Conservation of Momentum'

     on the basis of a single principle.

➡ Symmetry ⟷ Conservation Laws

➡ Governing principle: Variational principle –
Principle of Least Action

These methods have a charm of their own and very many applications….

Applications of Lagrange's/Hamilton's Equations

- Constraints / Degrees of Freedom
  - offers great convenience!

- '*Action*' : dimensions
  'angular momentum' :

$h:$ $Max\ Planck:$
$fundamental\ quantity$
$in\ Quantum\ Mechanics$

Illustrations: use of Lagrange's / Hamilton's equations

to solve simple problems in Mechanics

Manifestation of simple phenomena in different unrelated situations

Dynamics of

spring–mass systems,

➢ pendulum,

➢ oscillatory electromagnetic circuits,

➢ bio rhythms,

➢ share market fluctuations …

radiation oscillators,

molecular vibrations,

atomic, molecular, solid

state, nuclear physics,

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electrical engineering,

mechanical

engineering …

Musical instruments

SMALL OSCILLATIONS

1581:

Observations on the swaying chandeliers at the Pisa cathedral.

Galileo (when only 17 years old) recognized the constancy of the periodic time for small oscillations.

Use of Lagrange's / Hamilton's equations to solve the problem of Simple Harmonic Oscillator.

Generalized Coordinate

Generalized Velocity

Generalized Momentum

$$p = \left(\frac{\partial L}{\partial \dot{q}} \right)$$

$$H = H(q, p, t)$$

$$L = L(q, \dot{q}, t) \quad Lagrangian$$

2nd order

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \quad Lagrange's\ Equation$$

differential equation

$$H = \sum_k \dot{q}_k p_k - L$$

$$H = H(q, p, t) \qquad Hamiltonian$$

TWO

1st order

$$\forall\ k:\ \frac{\partial H}{\partial p_k} = \dot{q}_k\ \text{and}\ \frac{\partial H}{\partial q_k} = -\dot{p}_k$$

differential equations

Hamilton's Equations

Mass-Spring Simple Harmonic Oscillator

$$L = L(q, \dot{q}, t) \qquad \textit{Lagrangian}$$

$$L = T - V = \frac{m}{2} \dot{q}^2 - \frac{k}{2} q^2$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0$$

$$-kq - m\ddot{q} = 0 \qquad -\frac{k}{2} 2q - \frac{d}{dt} \left(\frac{m}{2} 2\dot{q} \right) = 0$$

$$m\ddot{q} = -kq$$

2nd order

differential equation

Newton's
Lagranges'

$$m\ddot{q} = -kq \qquad \ddot{q} = -\frac{k}{m}q$$

Linear relation between restoring force and displacement for spring-mass system:

Robert Hooke (1635-1703), (contemporary of Newton), empirically discovered this relation for several elastic materials in 1678.

$$\ddot{x} = -\frac{k}{m}x$$

HAMILTONIAN approach

$$L = L(q, \dot{q}, t)$$

Note! Begin

Always with the

LAGRANGIAN.

$$H = H(q, p, t)$$

VERY

IMPORTANT !

$$Lagrangian : L = T - V = \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2$$

$$p = \left(\frac{\partial L}{\partial \dot{q}} \right) = m\dot{q}$$

$$Lagrangian: L = T - V = \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2$$

$$p = \left(\frac{\partial L}{\partial \dot{q}}\right) = m\dot{q}$$

$$H = p\dot{q} - L = m\dot{q}^2 - \frac{m}{2}\dot{q}^2 + \frac{k}{2}q^2$$

$$H = \frac{p^2}{2m} + \frac{k}{2}q^2$$

$$H = \frac{p^2}{2m} + \frac{k}{2}q^2$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{2p}{2m} = \frac{p}{m}$$

and $\dot{p} = -\dfrac{\partial H}{\partial q_k} = -\dfrac{k}{2}2q$

$$(i.e. \quad f = -kq)$$

Hamilton's Equations
TWO first order equations

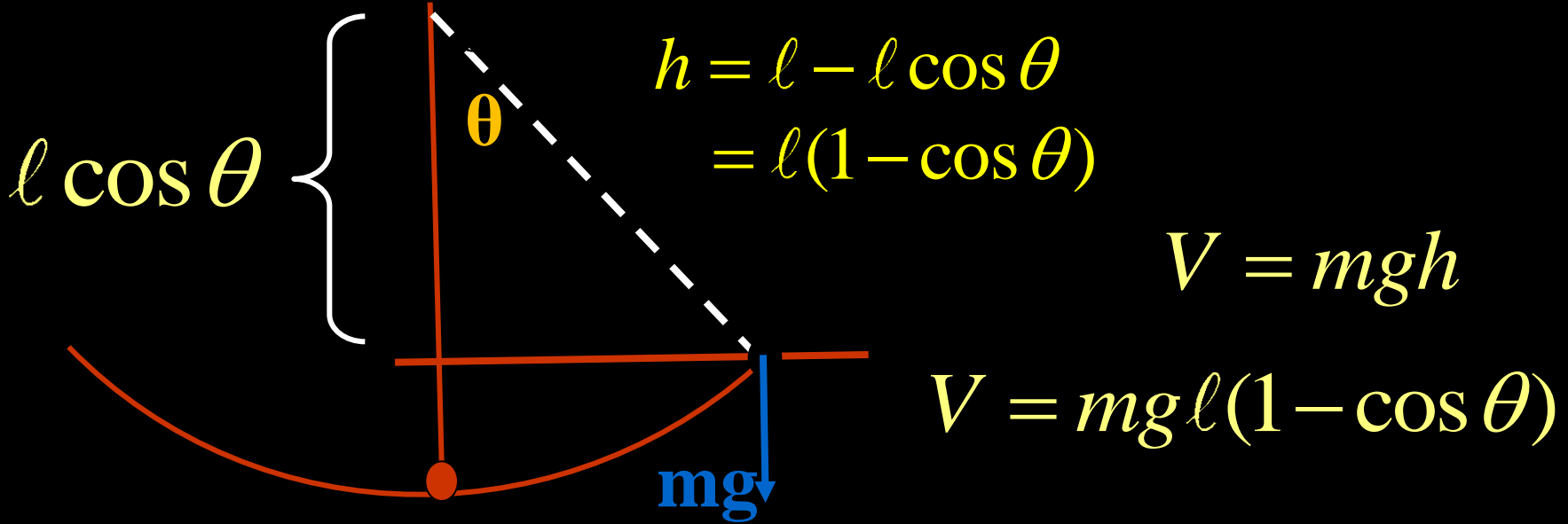Be careful about how you write the Lagrangian and the Hamiltonian for the Harmonic oscillator!

$$Lagrangian : L = T - V = \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2$$

$$L = L(q, \dot{q}, t)$$

$$p = \left(\frac{\partial L}{\partial \dot{q}}\right) = m\dot{q}$$

$$H = \frac{p^2}{2m} + \frac{k}{2}q^2$$

$$H = H(q, p, t)$$

VERY IMPORTANT !

Generalized Momentum is interpreted only as $p = \left(\frac{\partial L}{\partial \dot{q}}\right)$, and not a product of mass with velocity

$\ell\cos\theta$

θ

$h = \ell - \ell\cos\theta$
$= \ell(1-\cos\theta)$

$V = mgh$

$V = mg\ell(1-\cos\theta)$

mg

Remember this!
First: set-up the Lagrangian

$r=l$: *constant*

$L = L(r,\theta,\dot{r},\dot{\theta})$

$L = T - V$

$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mg\ell(1-\cos\theta)$

$L = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell(1-\cos\theta)$

$L = \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell + mg\ell\cos\theta$

$$L = \frac{1}{2} m\ell^2 \dot{\theta}^2 - mg\ell + mg\ell \cos\theta$$

$$r = \ell : \quad \text{fixed length}$$

Now, we can find the generalized momentum for each degree of freedom.

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\frac{\partial L}{\partial \dot{r}} = p_r = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = p_\theta = ml^2\dot{\theta}$$

Simple pendulum

$$L = \frac{1}{2} m\ell^2 \dot{\theta}^2 - mg\ell + mg\ell \cos\theta$$

$$\frac{\partial L}{\partial r} = 0.$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta \approx -mgl\theta$$

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\frac{\partial L}{\partial \dot{r}} = p_r = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = p_\theta = ml^2\dot{\theta}$$

$$-mgl\theta - \frac{d}{dt}(ml^2\dot{\theta}) = 0$$

$$-mgl\theta = ml^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{g}{l}\theta$$

$$\ddot{\theta} = -\frac{g}{l}\theta$$

$$
\begin{array}{ll}
(1) & \ddot{q} = -\alpha q \\
(2) & \text{Solution: } q = Ae^{i\omega_0 t} + Be^{-i\omega_0 t} \\
& \text{Substitute (2) in (1)} \Rightarrow \omega_0 = \sqrt{\alpha}
\end{array}
$$

$$\omega_0 = \sqrt{\frac{g}{l}}$$

(1) Newtonian
(2) Lagrangian

$$\ddot{\theta} = -\frac{g}{l}\theta$$

$$\omega_0 = \sqrt{\frac{g}{l}}$$

Note! We have not used 'force', 'tension in the string' etc. in the Lagrangian and Hamiltonian approach!

Hamilton's equations: simple pendulum

$$L = L(r, \theta, \dot{r}, \dot{\theta}) = T - V$$

$$L = \left(\frac{1}{2} m\ell^2 \dot{\theta}^2 \right) - \left(mg\ell - mg\ell \cos\theta \right)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = 0$$

$$H = \left[\sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right] = \left[\sum \dot{q}_i p_i - L \right]$$

$$H = \left[\sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L \right]$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta}$$

$$= \left[\sum \dot{q}_i p_i - L \right]$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = 0$$

$$L = \left(\frac{1}{2} m\ell^2 \dot{\theta}^2 \right) - \left(mg\ell - mg\ell \cos\theta \right)$$

$$H = \dot{\theta} p_\theta + \dot{r} p_r - \frac{1}{2} m\ell^2 \dot{\theta}^2 + mg\ell + mg\ell \cos\theta$$

$$\frac{\partial H}{\partial \theta} = mgl(-\sin\theta) \approx -mgl\theta$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = mgl\theta$$

simple pendulum

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = -ml^2\dot\theta$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = mgl\theta$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = 0$$

$$-ml^2\ddot\theta = \dot{p}_\theta = mgl\theta$$

$$\ddot\theta = -\frac{g}{l}\theta$$

(1) $\quad \ddot{q} = -\alpha q$

(2) Solution: $q = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$

Substitute (2) in (1) $\Rightarrow \omega_0 = \sqrt{\alpha}$

$$\omega_0 = \sqrt{\frac{g}{l}}$$

(1) Newtonian
  (2) Lagrangian
    (3) Hamiltonian

$$\ddot{\theta} = -\frac{g}{l}\theta$$

$$\omega_0 = \sqrt{\frac{g}{l}}$$

We have NOT used 'force',

causality , linear-response

Lagrangian and Hamiltonian Mechanics has very many applications.

All problems in 'classical mechanics' can be addressed using these techniques.

However, they do depend on the premise that

mechanical system is characterized by position and velocity/momentum,

simultaneously and accurately.

Central problem in 'Mechanics':

How is the 'mechanical state' of a system described, and how does this 'state' evolve with time?

- Formulations due to Galileo/Newton,
- Lagrange and Hamilton.

(q,p) : How do we get these?

Heisenberg's

principle of uncertainty

New approach required !

'New approach' is not required on account of the Heisenberg principle!

Rather,

the measurements of q and p are not compatible….

….. so how could one describe the mechanical state of a system by (q,p) ?

Heisenberg principle comes into play as a result of the fact that simultaneous measurements of q and p do not provide consistent accurate values *on repeated measurements*. ….

….. so how could one describe the mechanical state of a system by (q,p) ?

$| \quad \rangle$ Mechanical State:
State vectors in Hilbert Space

Characterize? Labels?

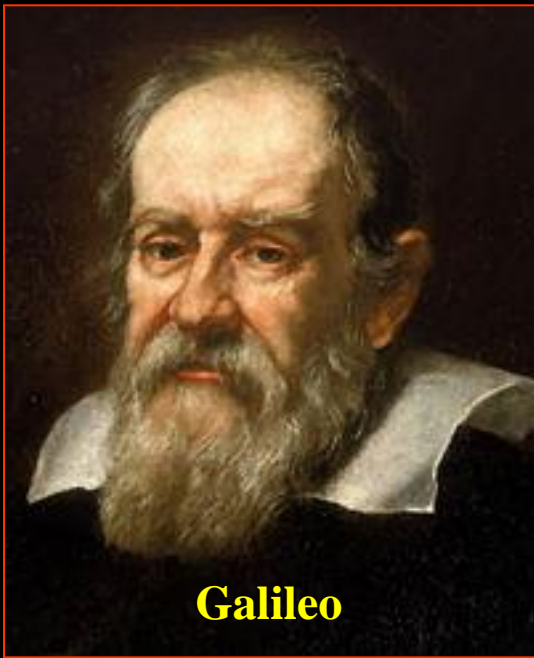"Good" quantum
numbers/labels

$| \text{Measurment: C.S.C.O. } \rangle$

Complete Set of Commuting Operators
Complete Set of Compatible Observables

$$i\hbar \frac{\partial}{\partial t} | \quad \rangle = H | \quad \rangle$$

Evolution of the
Mechanical
State of the system

Schrödinger Equation

Galileo

Newton

Lagrange

Hamilton

$(q, \dot{q})$

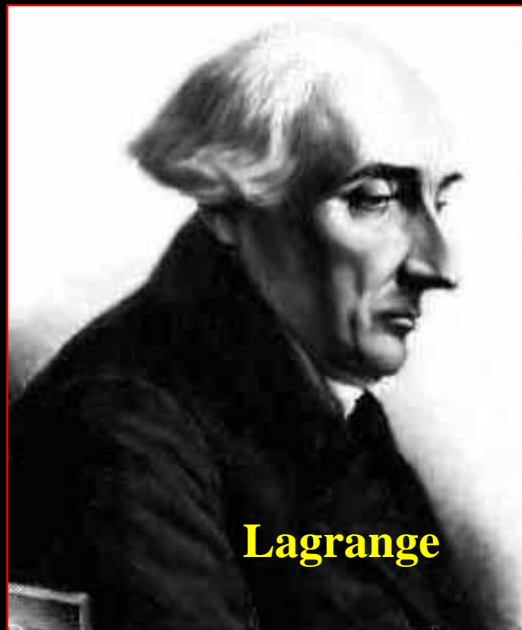$\vec{F} = m\vec{a}$

Linear Response.

Principle of causality.

Principle of
 Variation

$L(q, \dot{q})$

$H(q, p)$

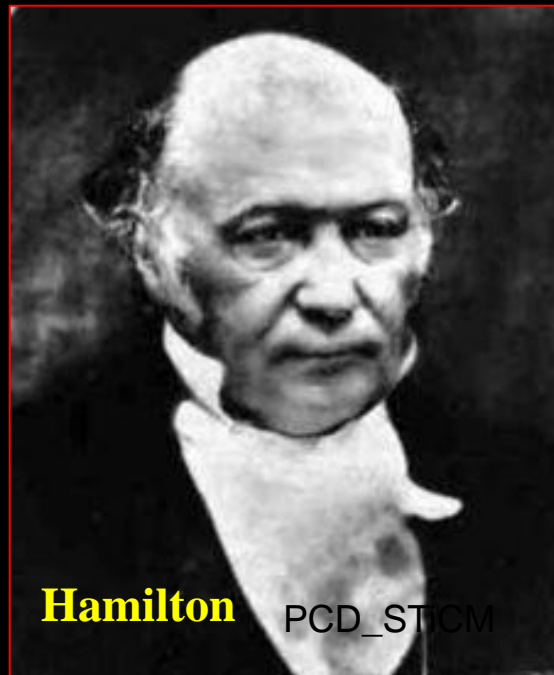$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q_k}$$

PCD_STICM