STiCM

Select / Special Topics in Classical Mechanics

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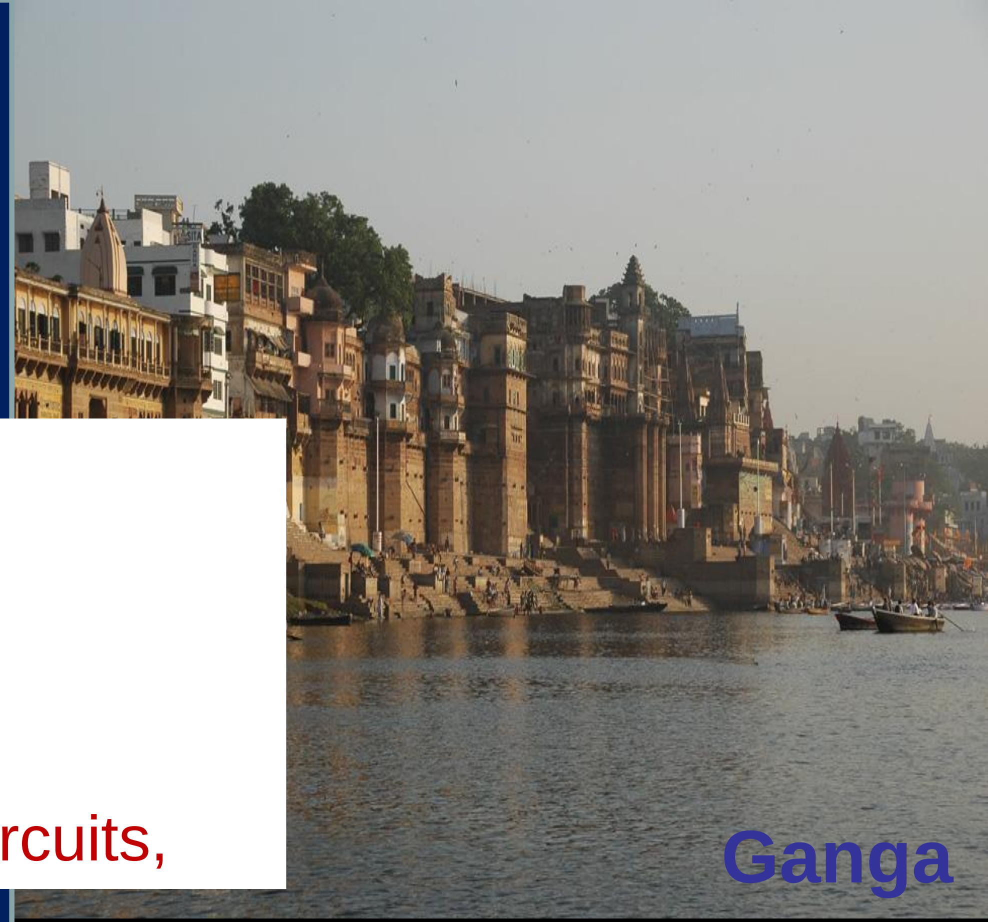
STiCM Lecture 07: Unit 2 Oscillators, Resonances, Waves

Oscillation: Repetitive Physical phenomenon

Dynamics of

spring–mass systems,

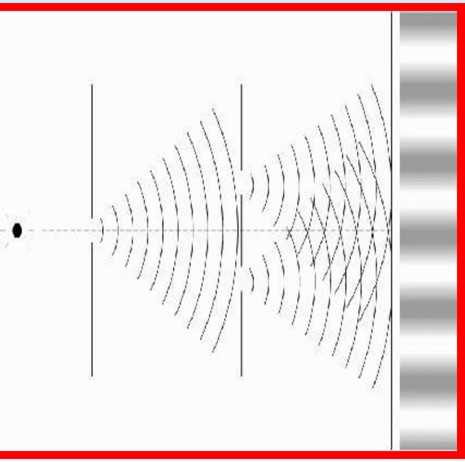
- pendulum,
- oscillatory electromagnetic circuits,
- bio rhythms,
- share market fluctuations

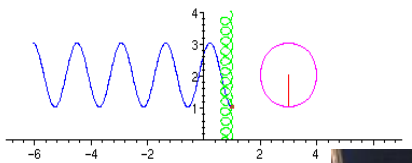


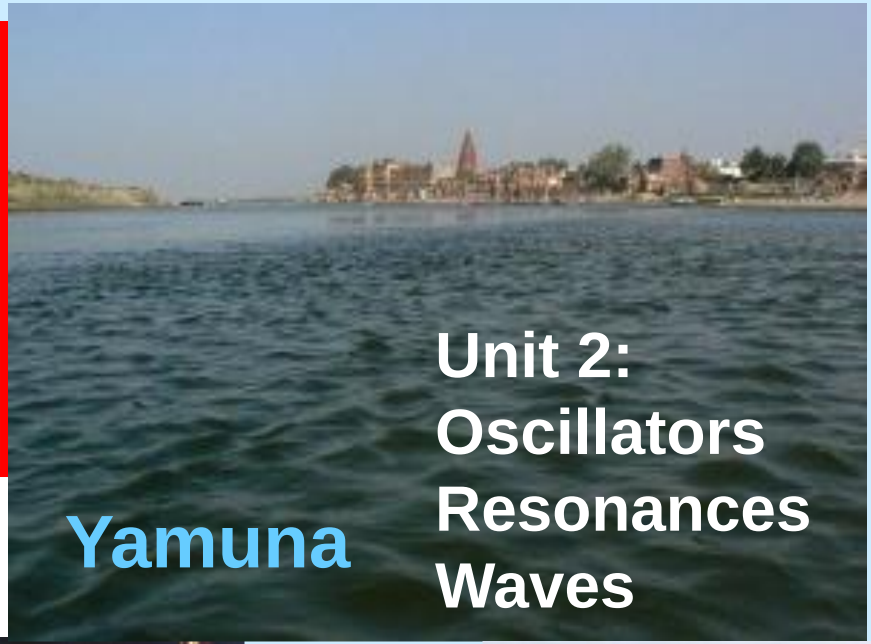
- radiation oscillators,

  molecular vibrations,

atomic, molecular, solid state, ……..







Unit 2: Oscillators Resonances Waves

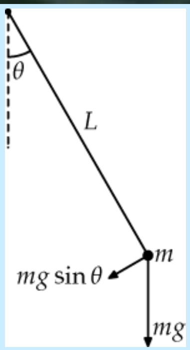
Oscillations.

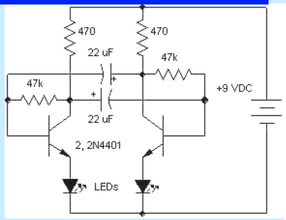
Small oscillations. SHM.

Driven and damped oscillator.

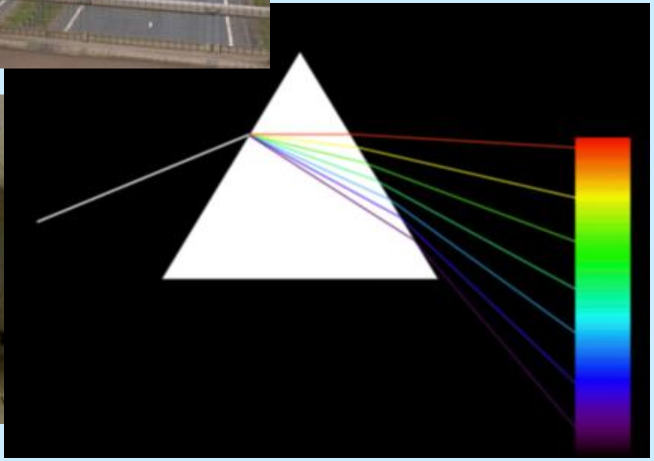
Resonance, Quality factor.

Waves.





Flip-Flop Square Wave oscillator



Learning Goals

- Recognize stable, unstable, neutral equilibrium points and saddle points.

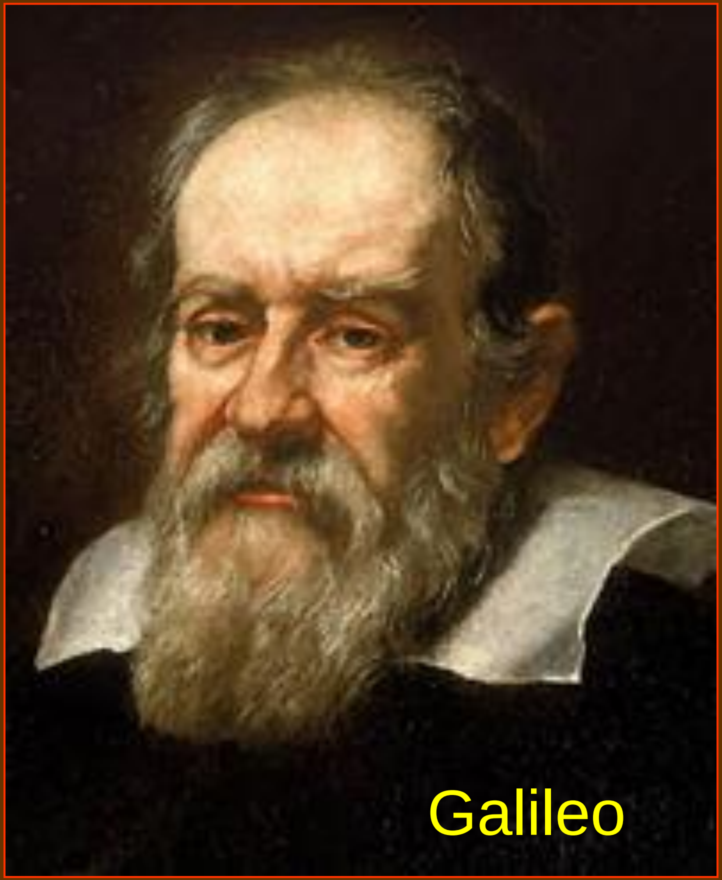
- Learn that in a region close enough to any point of stable equilibrium, motion can be described by the simple harmonic oscillator.

- Discover electro-mechanical analogies and how they can be exploited in solving problems in different branches of Physics. Learn about effects of damping, and effects of a periodic driving force.

Learning Goals

- Get introduced to resonances in physical systems and the primary indicators of the quality of measurement techniques, such as the 'Quality Factor'.

- We shall also become familiar with the 'wave motion' which is of ubiquitous application in both 'classical' and 'quantum mechanics'.

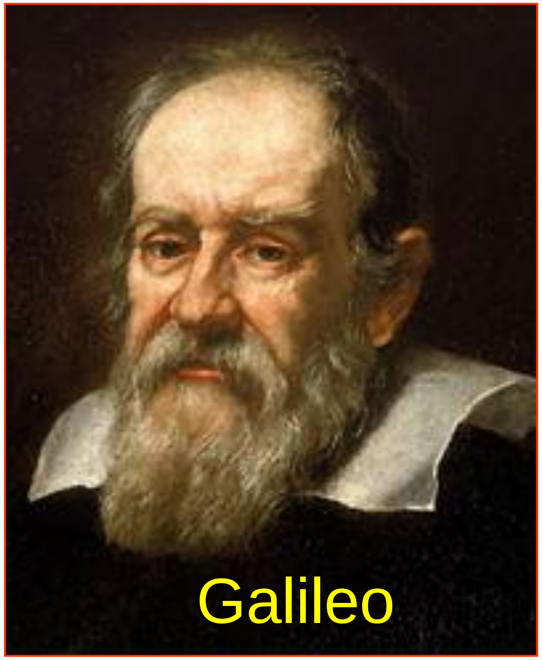


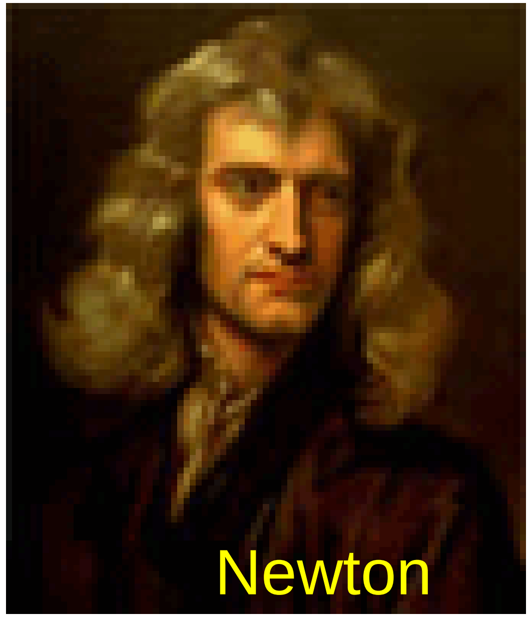
$$(q, \dot{q})$$

$$\vec{F} = m\vec{a}$$

Linear Response.

Principle of causality.





$(q, \dot{q})$

$\vec{F} = m\vec{a}$

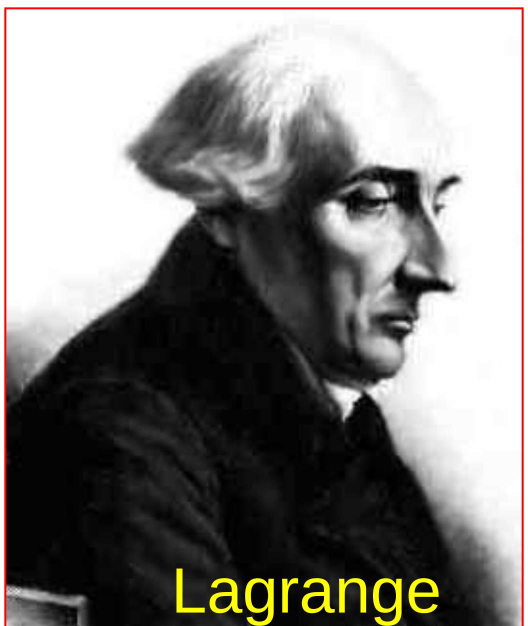
Linear Response.

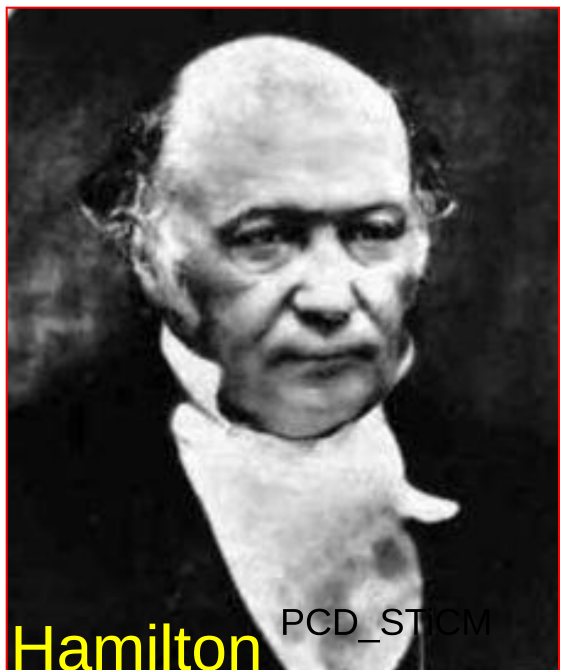
Principle of causality.

Principle of
Variation

$L(q, \dot{q})$

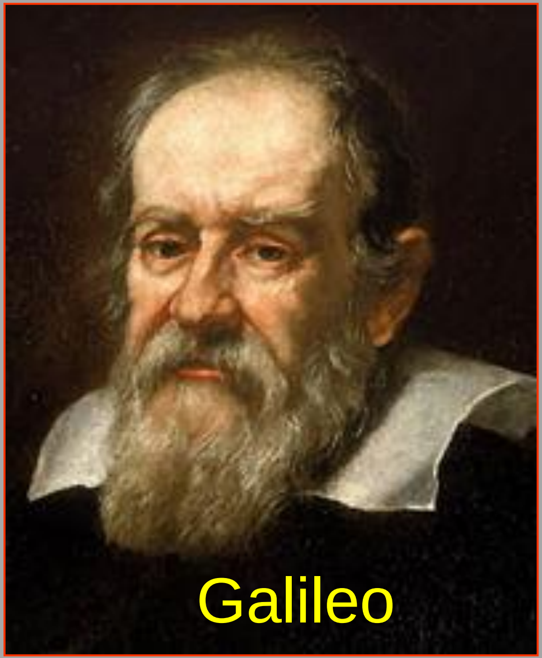
$H(q, p)$

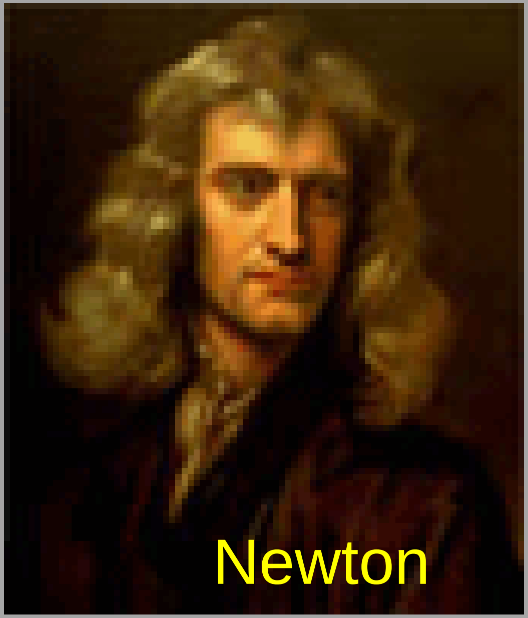




$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$$\dot{q} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial q_k}$$





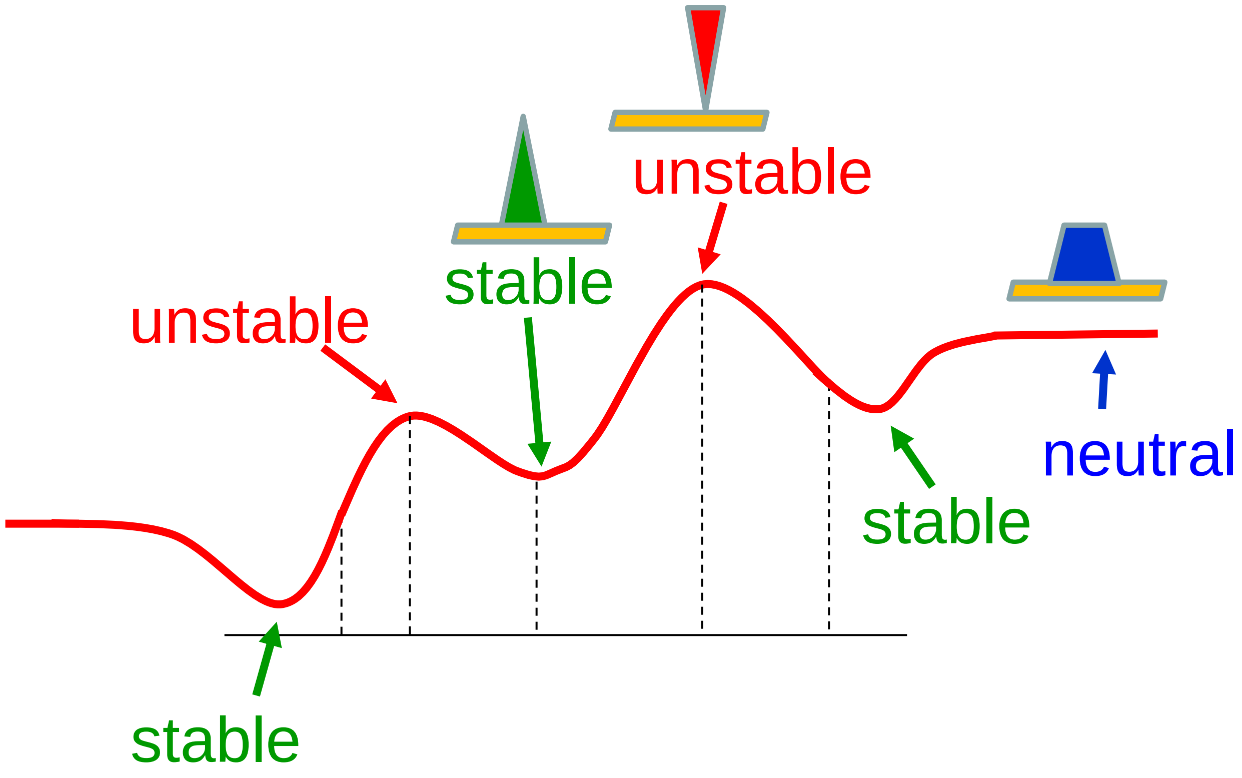
$(q, \dot{q})$

$\vec{F} = m\vec{a}$

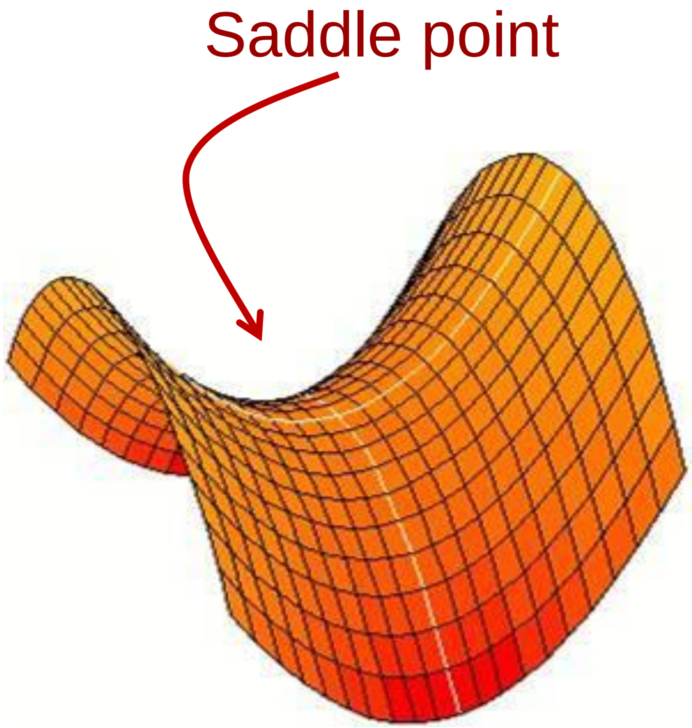
Linear Response.

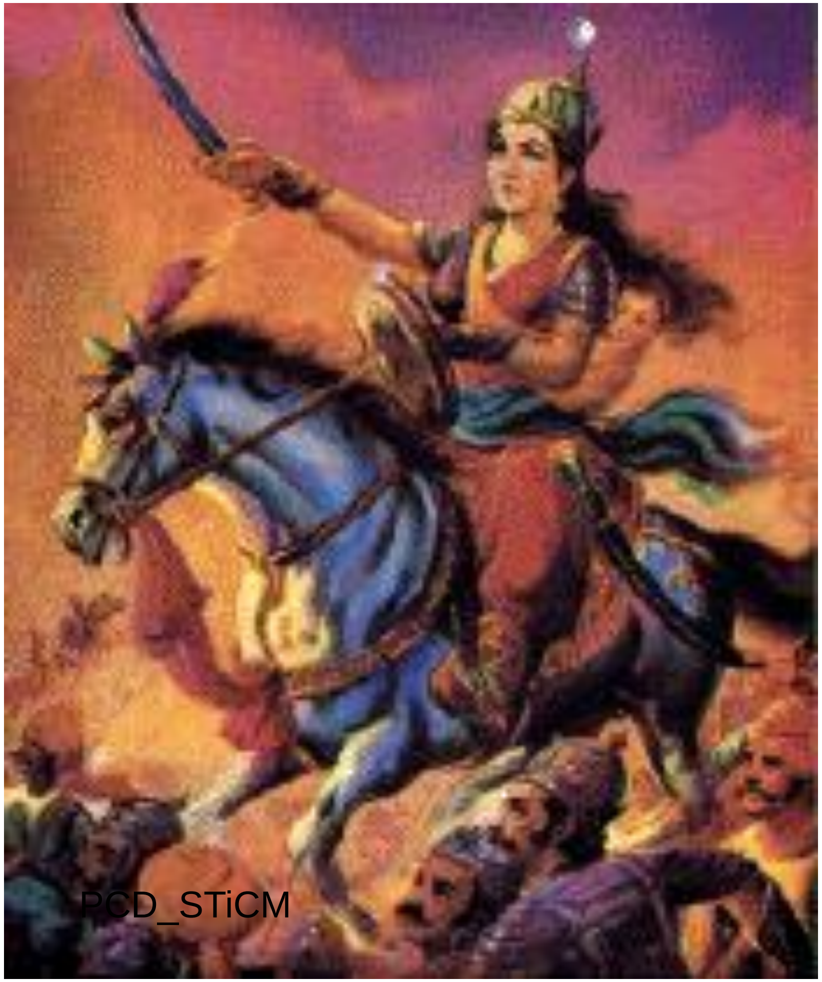
Principle of causality.

Kinds of equilibrium

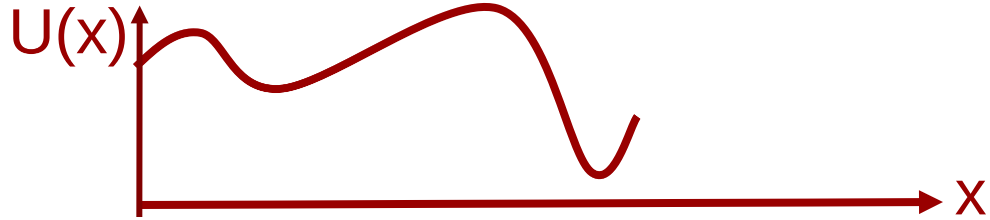


Un/stable equilibrium?





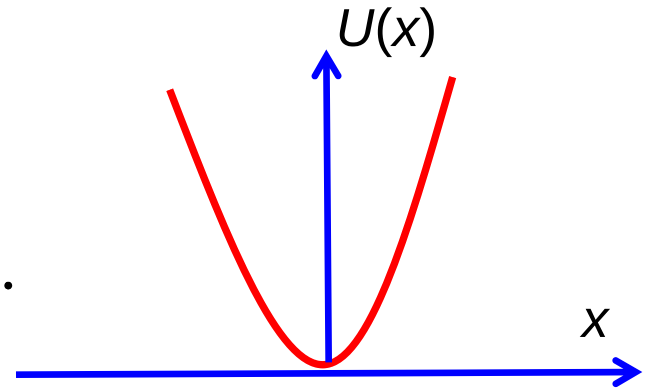
meaning of small oscillations



'Zero', at equilibrium

$$U(x) = U(x_0) + \left.\frac{\partial U}{\partial x}\right|_{x_0}(x - x_0) + \frac{1}{2!}\left.\frac{\partial^2 U}{\partial x^2}\right|_{x_0}(x-x_0)^2 + \frac{1}{3!}\left.\frac{\partial^3 U}{\partial x^3}\right|_{x_0}(x-x_0)^3 + \ldots$$

Approximations, close to x_0



Potential for a Linear harmonic oscillator

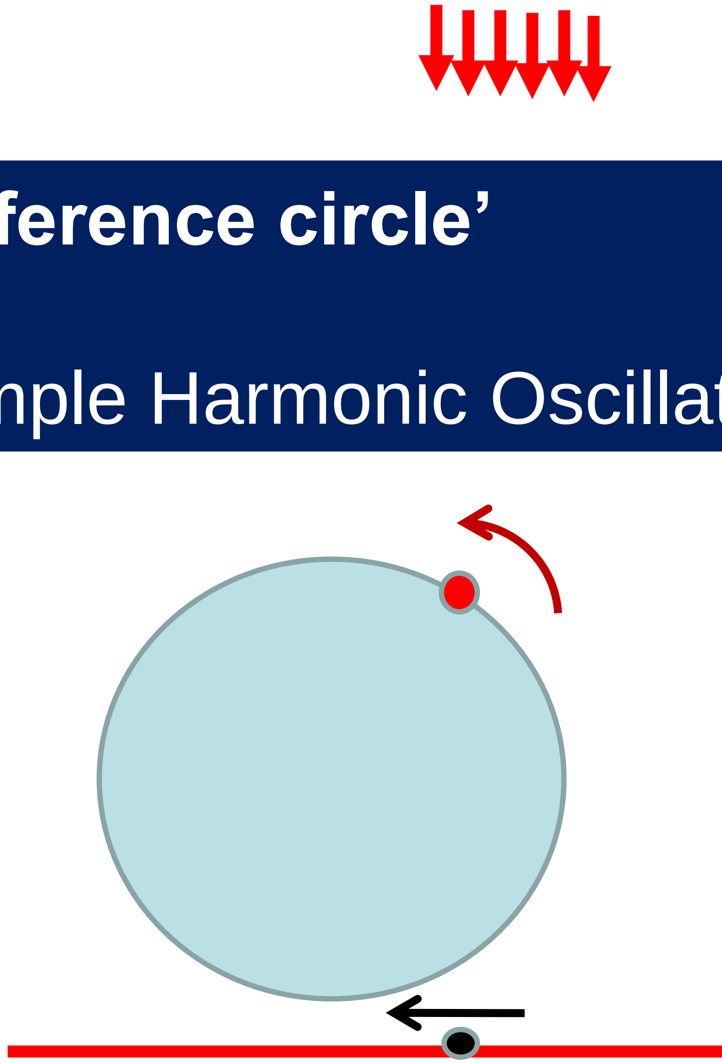
$$U(x) \approx U(x_0) + \frac{1}{2!}\left.\frac{\partial^2 U}{\partial x^2}\right|_{x_0}(x-x_0)^2 = \frac{1}{2}kx^2$$

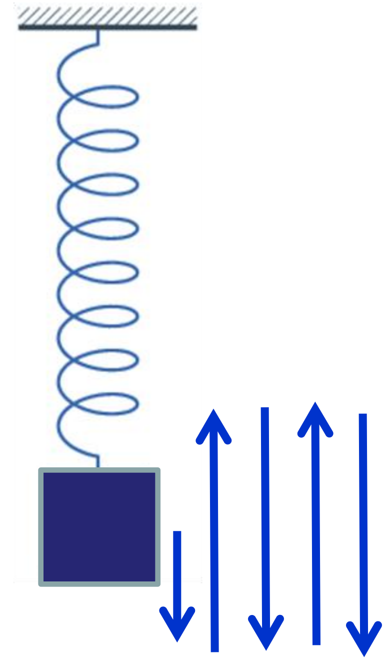
by choosing $U(x_0) = 0$ and $x_0 = 0$.

$$F = -\frac{dU}{dx} = -kx$$

$$\ddot{x} = -\frac{k}{m}x$$

‘reference circle’
for
Simple Harmonic Oscillations

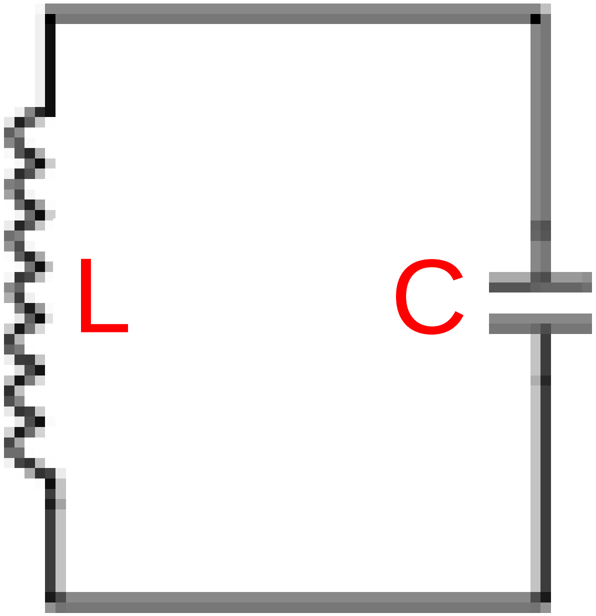




Intrinsic natural frequency

$$\omega = 2\pi\nu$$

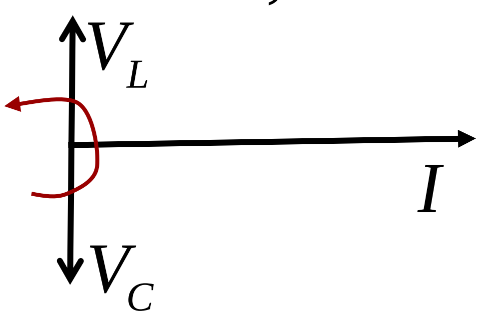
Shadow of the red dot in uniform circular motion constitutes SHM



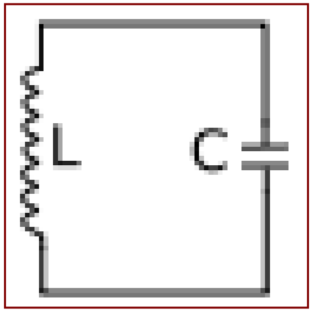
Unlike what happens in a resistor, the current and voltage in an inductance L, and in a capacitor C, does not peak together.

I is proportional to $\frac{dV}{dt}$,

not to V, as in the case of a resistor.



Voltage lags the current in a capacitor by 90^0,
but
leads the current in an inductor by the same amount.



$$V_C = \frac{Q}{C}$$

$$V_L = -L\frac{dI}{dt} = -L\frac{d^2Q}{dt^2} = -L\ddot{Q}$$

$$-V_L + V_C = 0$$

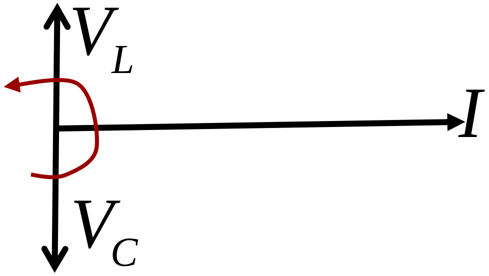
$$+L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

$$\ddot{Q} = -\left(\frac{1}{LC}\right)Q$$

$$I = \dot{Q} = \frac{d}{dt}Q = \frac{d}{dt}(CV) = C\frac{dV}{dt}$$

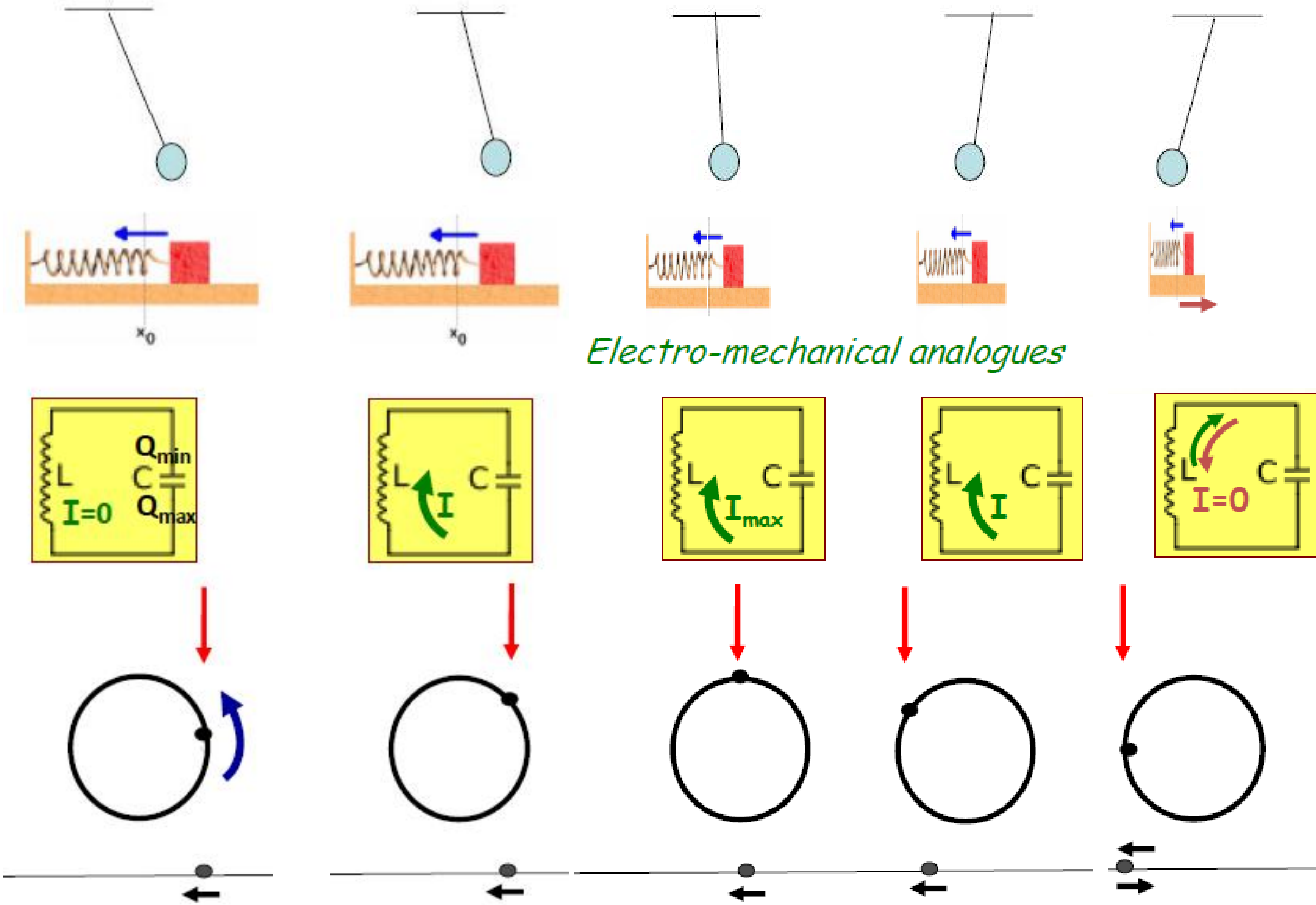
I is proportional to $\frac{dV}{dt}$,

not to V, as in the case of a resistor.



Voltage lags the current in a capacitor by 90^0,
but leads the current in an inductor by the same amount.

Electro-mechanical analogues



$$\ddot{x} = -\frac{k}{m} x$$

$$\ddot{Q} = -\left(\frac{1}{L}\frac{1}{C}\right) Q$$

Electro-mechanical analogues:

Inductance ⟷ mass, inertia

Capacitance ⟷ 1/k, compliance

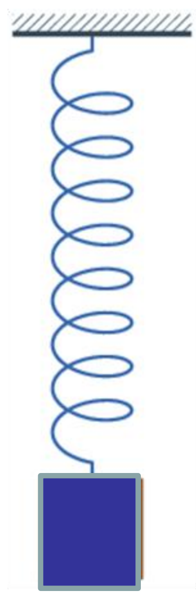
(1) $\ddot{q} = -\alpha q$

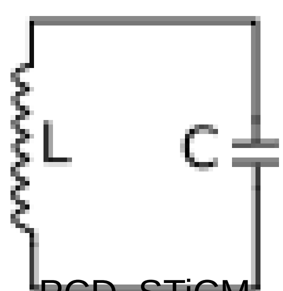
(2) Most general solution: $q = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$

Substitute (2) in (1) $\Rightarrow \omega_0 = \sqrt{\alpha}$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\omega_0 = \sqrt{\frac{1}{LC}}$$





A & B: determined by INITIAL CONDITIONS

Question:

Could we have associated L with 1/k and C with m?

$$\ddot{Q} = -\left(\frac{1}{LC}\right)Q$$ electrical LC circuit oscillator

(1) $\ddot{q} = -\alpha q$

(2) Most general solution: $q = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$

Any wonder that Feynman calls the above relation as Newton's law of electricity' ?

Two initial conditions provide solutions to the 'equation of motion' in a linear response formalism.

$$\ddot{q} = -\alpha q(t)$$

The most general solution is

$$q(t) = Ae^{i\omega_0 t} + \mathrm{B}e^{-i\omega_0 t} \text{ where } \omega_0 = \sqrt{\alpha}$$

the frequency is governed by α;

A and B are determined by initial conditions

the solution at time $t = 0$ is

$q(t=0) = A + B$; also, $\dot{q}(t=0) = i\omega_0 (A - B)$

solving for A an B from the two equations,

$$A = \frac{1}{2} \left\{ q(t=0) - i\frac{\dot{q}(t=0)}{\omega_0} \right\};$$

$$B = A^* \text{ (complex conjugate)}$$

$$\ddot{x} = -\frac{k}{m}x$$

spring-mass system

$$x = A\sin(\omega_0 t + \varphi)$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Mean: $\langle f(t) \rangle = \dfrac{\int_{t=0}^{T} f(t)dt}{\int_{t=0}^{T} dt}$

Mean kinetic energy

$$\langle KE \rangle = \left\langle \frac{1}{2} m\dot{x}^2 \right\rangle$$

Mean potential energy

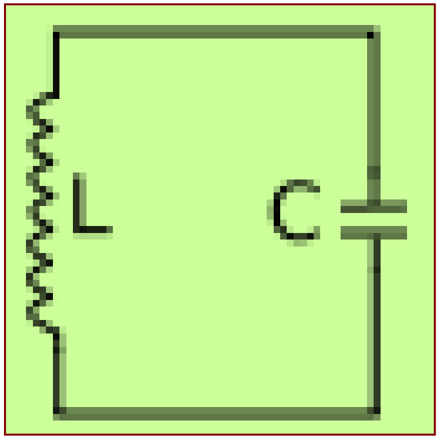
$$\langle PE \rangle = \left\langle \frac{1}{2} kx^2 \right\rangle$$

$$<\cos^2(\omega t)dt> = \frac{1}{2} = <\sin^2(\omega t)dt>$$

$$\langle KE \rangle = \frac{1}{2} m \langle \dot{x}^2 \rangle = \frac{1}{2} m(A\omega_0)^2 < \frac{1}{2} >= \frac{1}{4} A^2 m \omega_0{}^2$$

$$\langle PE \rangle = \left\langle \frac{1}{2} kx^2 \right\rangle = \frac{1}{2} kA^2 < \frac{1}{2} >= \frac{1}{4} kA^2$$

$$\textit{Note} :< PE >=< KE >, \text{ since } \omega_0{}^2 = \frac{k}{m}$$



$$\ddot{Q} = -\left(\frac{1}{LC}\right)Q$$

(1) $\ddot{q} = -\alpha q$

(2) Most general solution: $q = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$

Substitute (2) in (1) $\Rightarrow \omega_0 = \sqrt{\alpha}$

Graph plotting exercises

a) plot q and $\dot{q}$ as functions of t

b) sketch instantaneous V and I as functions of t

c) what is the phase difference between q and $\dot{q}$?

d) what is the phase difference between I and V ?

(1) $\ddot{q} = -\alpha q$

(2) Most general solution: $q = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$

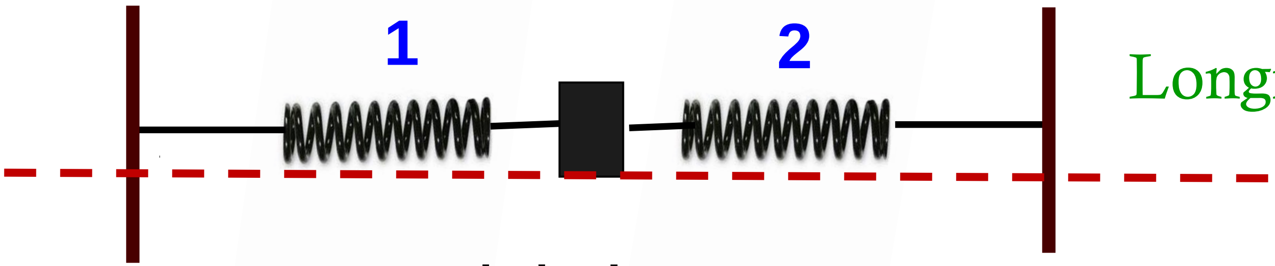
$\omega_0 = \sqrt{\alpha}$

SUPERPOSITION

COUPLED OSCILLATORS

Longitudinal oscillations

Reference: Berkeley's Mechanics



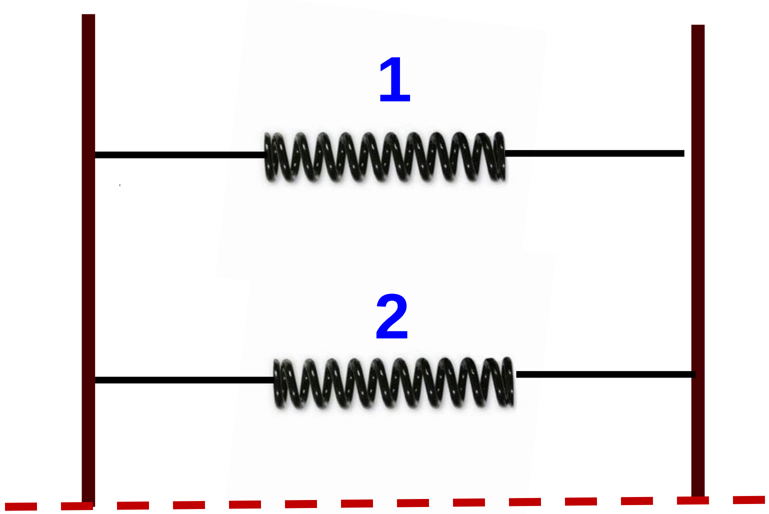
Frictionless support

Longitudinal Oscillations

Coupled Oscillators

Principle of superposition

Frequency of oscillations?

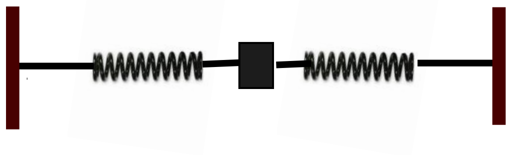


Frictionless support

Longitudinal Displacement, to the left or right, both make BOTH THE SPRINGS apply a restoring force on the mass in essentially THE SAME DIRECTION.

'effective spring constant' = ?

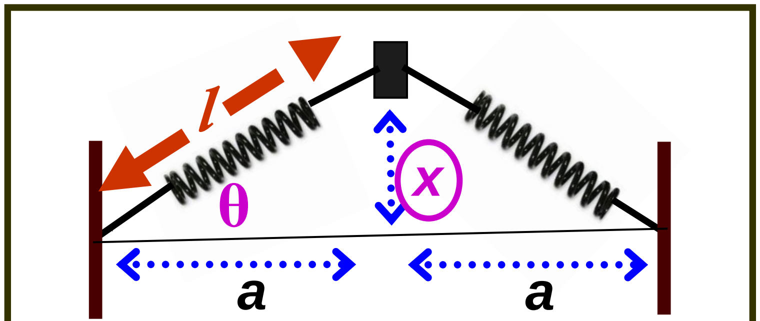
a_o: relaxed length of the springs
a: instantaneous stretched length



Tension exerted by each string

AT EQUILIBRIUM

$$T = k(a - a_0)$$



View in the plane of vibration

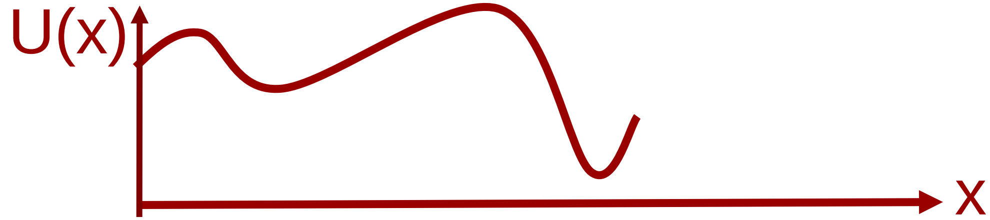
Transverse oscillations

The total restoring force along $-x$

is $-2T\sin\theta$

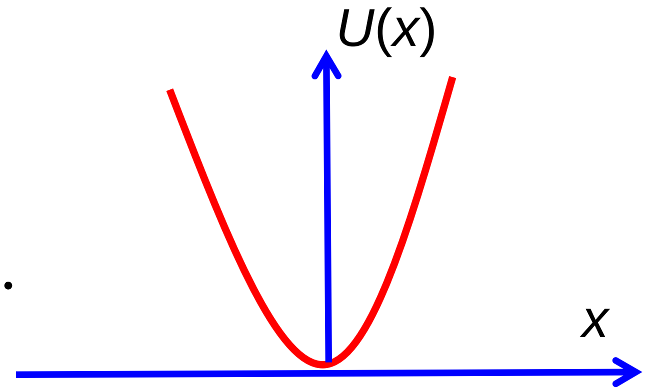
$$m\ddot{x} = -2T\sin\theta = -2k(l - a_0)\frac{x}{l}$$

meaning of small oscillations



$$U(x) = U(x_0) + \left.\frac{\partial U}{\partial x}\right|_{x_0} (x - x_0) + \left.\frac{1}{2!}\frac{\partial^2 U}{\partial x^2}\right|_{x_0} (x - x_0)^2 + \left.\frac{1}{3!}\frac{\partial^3 U}{\partial x^3}\right|_{x_0} (x - x_0)^3 + \ldots$$

Approximations, close to x_0



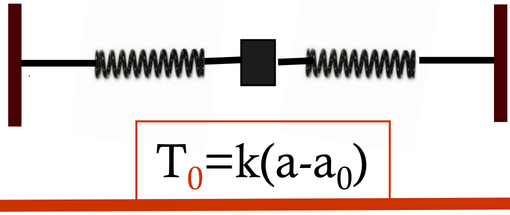
Potential for a Linear harmonic oscillator

$$F = -\frac{dU}{dx} = -kx$$

$$\ddot{x} = -\frac{k}{m}x$$

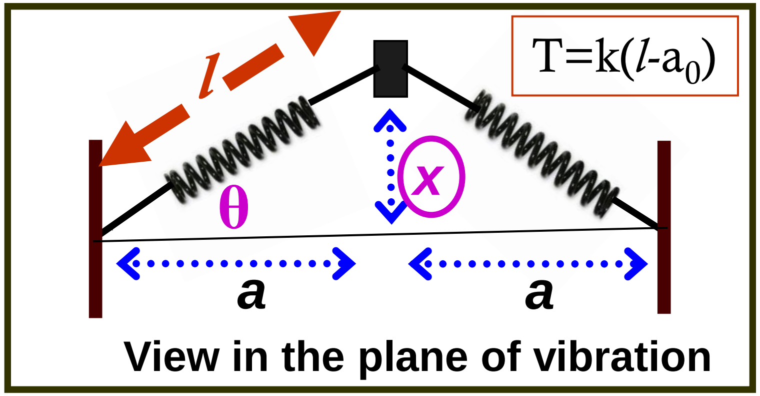
$$U(x) \approx U(x_0) + \left.\frac{1}{2!}\frac{\partial^2 U}{\partial x^2}\right|_{x_0} (x - x_0)^2 = \frac{1}{2}kx^2$$

by choosing $U(x_0) = 0$ and $x_0 = 0$.



a_o: relaxed length of the springs

a: instantaneous stretched length



Ref.: Berkeley, Vol.1/Mechanics

$$m\ddot{x} = -2T\sin\theta = -2k(l-a_0)\frac{x}{l}$$

SLINKY approximation

if $a_0 <<< l$ i.e. $\frac{a_0}{l} <<< 1$; $\frac{(l-a_0)}{l} \approx 1$; $\ddot{x} \approx -2\frac{k}{m}x$

SLINKY ~ SHO with effective spring constant (2k),

- for very large values of ***l*** without losing linear elasticity!

A typical slinky with a_0 of only 3” can be stretched to as much as ~15’ without loosing the linear elasticity!

(1) $\ddot{q} = -\alpha q$

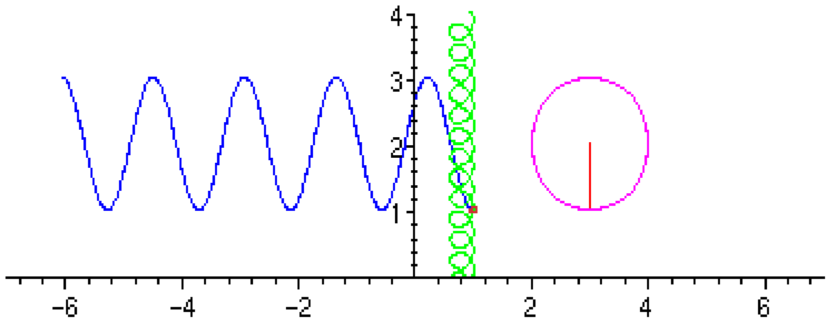
(2) Gen. solution: $q = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$

Substitute (2) in (1) $\Rightarrow \omega_0 = \sqrt{\alpha}$

Displacement $q(t=0) = A + B$

$\dot{q}(t=0) = i\omega_0 (A - B)$

We can find A and B in terms of $q(t=0)$ *and* $\dot{q}(t=0)$



Displacement $q = q(x,t) = q_0 \cos \omega t$

However, $q_0 \cos\left(\omega t \pm \phi(x)\right)$

is also a solution

What is the functional form of $\phi(x)$?

over one wavelength, ϕ must change through 2π

$$\frac{\partial \phi}{\partial x} = \frac{2\pi}{\lambda} \text{ and } \phi = \frac{2\pi}{\lambda} x + \Delta = kx + \Delta$$

where Δ is some constant angle.

$$q(t) = q_0 \cos\left(\omega t \pm \phi(x)\right) \qquad \phi = \frac{2\pi}{\lambda} x + \Delta = kx + \Delta$$

$$q(t) = q_0 \cos\left\{\omega t \pm (kx + \Delta)\right\} = \mathrm{q_0 cos\{}\omega \mathrm{t \pm kx \pm \Delta\}}$$

$$phase: \ \theta = \omega t \pm kx \pm \Delta$$

On a surface of constant *phase*: $d\theta = 0$

$$i.e. \ \omega dt \pm k dx = 0$$

$$i.e. \quad k dx = \mp \omega dt$$

$$\frac{dx}{dt} = \mp \frac{\omega}{k}$$

$$-\infty \longleftarrow \quad 0 \quad \longrightarrow +\infty$$

when

$$\frac{\mathrm{dx}}{\mathrm{dt}} < 0$$

$\leftarrow$ a wave travelling to the left

when

$$\frac{\mathrm{dx}}{\mathrm{dt}} > 0$$

$\rightarrow$ a wave travelling to the right

f(x-vt) represents a pulse traveling to the right

$$\frac{dx}{dt} > 0, \text{ i.e. } \frac{dx}{dt} \text{ as a positive quantity}$$

$\rightarrow$ a wave travelling to the right

---

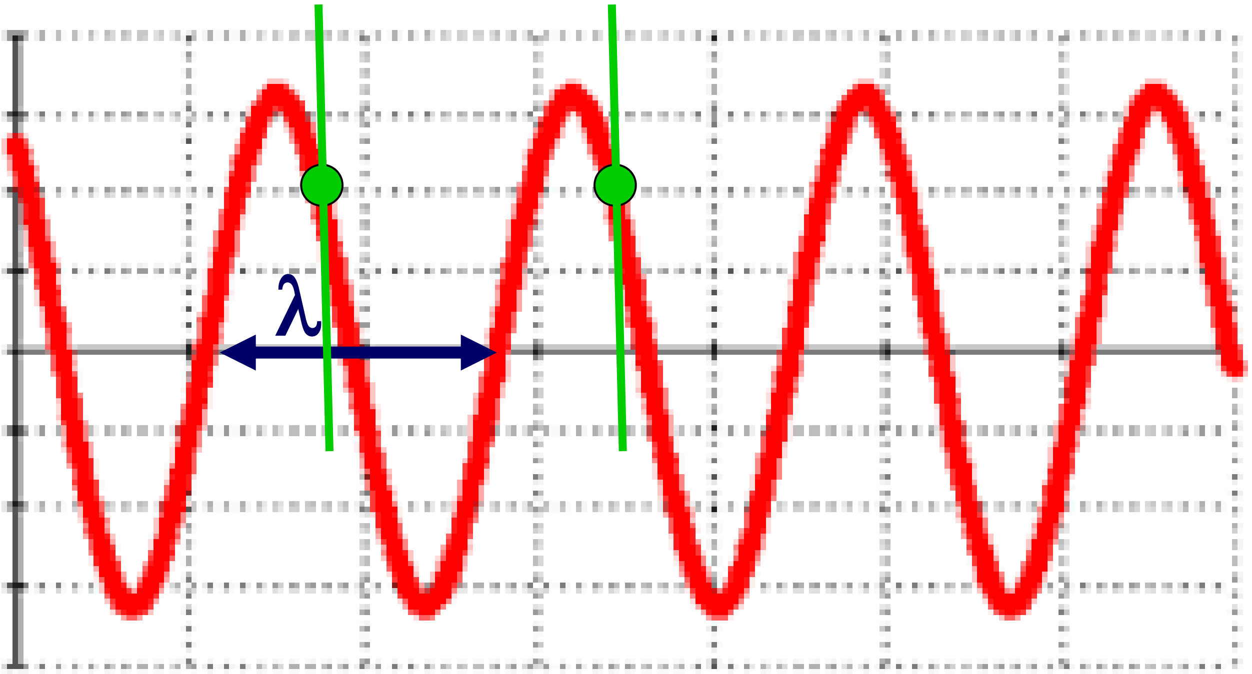
g(x+vt) represents a pulse traveling to the left

$$\frac{dx}{dt} < 0, \text{ i.e. } \frac{dx}{dt} \text{ as a negative quantity}$$

$\leftarrow$ a wave travelling to the left

The wave covers one wavelength λ in one period T,

The traveling speed of the wave is v= $\dfrac{\lambda}{T} = \nu\lambda$


λ

We will take a break….

……Any Questions ?

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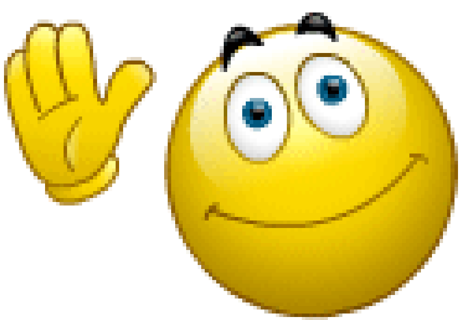

Next: DAMPED oscillations

STiCM

Select / Special Topics in Classical Mechanics

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STiCM Lecture 08: Unit 2 Oscillators, Resonances, Waves

Total energy E is constant: conservative forces

<KE> = <PE> : not true when friction is present

Damped harmonic oscillator

Is there only a restoring force in real situations?

Energy dissipation

Breaking, damping in automobiles,

galvanometer

S.H.O. $m\ddot{x} = -kx$

where $\omega_0{}^2 = \dfrac{k}{m}$

Damped Oscillator:

$F_{friction} = -c\text{v} = -c\dot{x}$

$$m\ddot{x} = -kx - c\dot{x}$$

$$\Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0{}^2 x = 0$$

$$\gamma = \frac{c}{2m}$$

If EM & Gravitational forces are conservative,

and all forces are made up of fundamental forces,

Then, why is friction dissipative?

Just what is ‘lost’, and why?

All ‘net’ interactions in nature:

superpositions of fundamental interactions,

- nuclear (‘strong’ interaction),
- electro-weak
  (electromagnetic/nuclear ‘weak’),
- and gravity.

So, what is the origin of dissipation?

Cause of ‘Friction’: Often, we track the evolution of the state of some pre-specified mechanical system without keeping track of everything else that this system interacts with.

There are thus unspecified degrees of freedom !

Dissipation: result of our neglect of these unspecified degrees of freedom,
even as the component interactions individually conserve energy.

The equation of motion: $m\ddot{x} = -kx - b\dot{x}$

$$\Rightarrow \ddot{x} + 2\gamma\dot{x} + \omega_0^{\ 2}x = 0$$ Eq.[a]

$$\text{where } \omega_0^{\ 2} = \frac{k}{m} \qquad \gamma = \frac{c}{2m}$$

We seek a solution in the form $x(t) = Ae^{qt}$ Eq.[b]

Why seek this form?

and inquire what conditions would result on *q* if Eq.[b] is to be admitted as a solution of Eq.[a]

Substitute [b] in [a]: $\dot{x}(t) = Aqe^{qt}, \quad \ddot{x}(t) = Aq^2e^{qt}$

$$Aq^2e^{qt} + 2\gamma Aqe^{qt} + \omega_0^{\ 2}Ae^{qt} = 0$$

$$q^2 + 2\gamma q + \omega_0^{\ 2} = 0$$

$$q^2 + \frac{c}{m} q + \omega_o^{\ 2} = 0$$

quadratic equation

$$m\omega_0^{\ 2} = k$$

$$mq^2 + cq + k = 0.$$

$$q = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

$$\gamma = \frac{c}{2m}$$

$$q_1 = -\gamma + \sqrt{\gamma^2 - \omega_o^{\ 2}},$$

$$q_2 = -\gamma - \sqrt{\gamma^2 - \omega_o^{\ 2}}$$

$$x(t) = A_1 e^{q_1 t} + A_2 e^{q_2 t};$$

A_1 and A_2 are constants

determined by initial conditions,

at $t = 0$, on $x(t)$, $\dot{x}(t)$

$$q_1 = -\gamma + \sqrt{\gamma^2 - \omega_o^{\;2}},$$

$$q_2 = -\gamma - \sqrt{\gamma^2 - \omega_o^{\;2}}$$

$$\omega_0^{\;2} = \frac{k}{m} \qquad \gamma = \frac{c}{2m}$$

CASE 1:

When $\gamma > \omega_0$,

OVERDAMPED OSCILLATOR

$\sqrt{\gamma^2 - \omega_o^{\;2}}$ is a real number whose value/magnitude is $< \gamma$,

so both q_1 and q_2 become 'real' and essentially 'negative'

$$Since: \quad x(t) = A_1 e^{q_1 t} + A_2 e^{q_2 t};$$

both the terms approach zero as $t \to \infty$, *asymptotically*

$$q_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_o^2},$$

$$x(t) = A_1 e^{q_1 t} + A_2 e^{q_2 t}$$

$$\omega_0^2 = \frac{k}{m}$$

$$= \frac{c}{2m}$$

When $\gamma > \omega_0$,

$\sqrt{\gamma^2 - \omega_o^2}$ is a real number whose value is $< \gamma$,

so both q_1 and q_2 become 'real' and essentially 'negative'

$$x(t) = A_1 e^{q_1 t} + A_2 e^{q_2 t}$$

$$\dot{x}(t) = q_1 A_1 e^{q_1 t} + q_2 A_2 e^{q_2 t}$$

Hence,

$$x(t=0) = A_1 + A_2$$

$$\dot{x}(t=0) = q_1 A_1 + q_2 A_2$$

$$q_1 x(t=0) = q_1 A_1 + q_1 A_2$$

$$\dot{x}(t=0) = q_1 A_1 + q_2 A_2$$

hence

$$\frac{q_1 x(t=0) - \dot{x}(t=0)}{q_1 - q_2} = A_2$$

$$x(t=0) - \frac{q_1 x(t=0) - \dot{x}(t=0)}{q_1 - q_2} = A_1$$

'Overshoot' : not possible. Oscillations being completely killed, this oscillator is called 'OVERDAMPED'.

$$\omega_0^{\,2} = \frac{k}{m}$$

$$\gamma = \frac{c}{2m}$$

CASE 2

UNDERDAMPED OSCILLATOR

$\gamma \langle \omega_0 \;\Rightarrow\; \sqrt{\gamma^2 - \omega_o^{\,2}}$ is an imaginary number

$$q_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_o^{\,2}} \,,$$

$$q_1 = -\gamma + \mathrm{i}\sqrt{\omega_0^{\,2} - \gamma^2} = -\gamma + i\omega \qquad where\ \omega = \sqrt{\omega_0^{\,2} - \gamma^2}$$

$$q_2 = -\gamma - \mathrm{i}\sqrt{\omega_0^{\,2} - \gamma^2} = -\gamma - i\omega \qquad i.e., \omega < \omega_0$$

by an amount determined by γ

$$x(t) = A_1 e^{(-\gamma + i\omega)t} + A_2 e^{(-\gamma - i\omega)t}$$

$$x(t) = e^{-\gamma t}\left\{A_1 e^{+i\omega t} + A_2 e^{-i\omega t}\right\}$$

$$x(t) = e^{-\gamma t}\left\{(A_1 + A_2)\cos(\omega t) + i(A_1 - A_2)\sin(\omega t)\right\}$$

CASE 2

UNDERDAMPED OSCILLATOR

When $\gamma < \omega_0, \sqrt{\gamma^2 - \omega_o^2}$ is an imaginary number

$$\omega_0^2 = \frac{k}{m}$$

$$\gamma = \frac{c}{2m}$$

$$x = e^{-\gamma t}\left\{A_1 e^{+i\omega t} + A_2 e^{-i\omega t}\right\}$$

$$x = e^{-\gamma t}\left\{(A_1 + A_2)\cos(\omega t) + i(A_1 - A_2)\sin(\omega t)\right\}$$

✔

Introduce two new parameters B & θ instead of A1 and A2.
→ insight in the nature of the solutions

$$A_1 + A_2 = B\sin\theta$$
$$i(A_1 - A_2) = B\cos\theta$$

$$A_1 = -\frac{iBe^{+i\theta}}{2}, \quad A_2 = +\frac{iBe^{-i\theta}}{2}$$

$$x(t) = Be^{-\gamma t}\left\{\sin\theta\cos(\omega t) + \cos\theta\sin(\omega t)\right\}$$

$$x(t) = Be^{-\gamma t}\sin(\omega t + \theta)$$

UNDERDAMPED OSCILLATOR

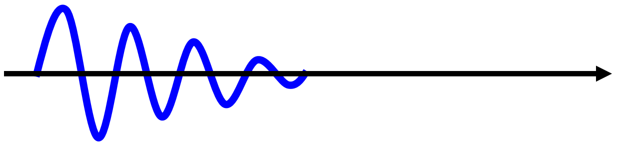
$\gamma < \omega_0, \sqrt{\gamma^2 - \omega_o{}^2}$:imaginary

$$\omega_0{}^2 = \frac{k}{m}$$

$$x(t) = Be^{-\gamma t}\left\{\sin\theta\cos(\omega t) + \cos\theta\sin(\omega t)\ \right\}$$

$$\gamma = \frac{c}{2m}$$

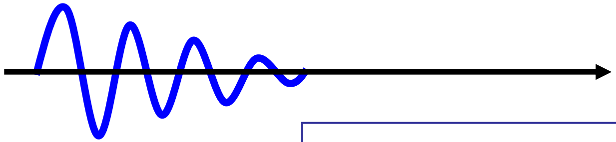
$$x(t) = Be^{-\gamma t}\sin(\omega t + \theta)$$



$$\omega = \sqrt{\omega_0{}^2 - \gamma^2}$$

$i.e., \omega < \omega_0$ by an amount determined by γ

- Solution: sinusoidal, at circular frequency ω determined by the two parameters ω_0 and γ.
- Frequency $\omega < \omega_0$
- Amplitude decreases exponentially with time
- Oscillation is phase shifted by θ



UNDERDAMPED OSCILLATOR

$$\omega_0^{\ 2} = \frac{k}{m}$$

$$\gamma = \frac{c}{2m}$$

When $\gamma < \omega_0$, $\sqrt{\gamma^2 - \omega_o^{\ 2}}$ is an imaginary number

$$x(t) = Be^{-\gamma t}\sin(\omega t + \theta)$$

This solution is *NOT* “periodic”; *NOT* repetitive.

One may regard the oscillatory sinusoidal term to have an *exponentially diminishing amplitude.*

But the *ZEROES* are repetitive; strictly periodic; occur at a time period of T=2π/ω, called “period of the damped oscillator”.

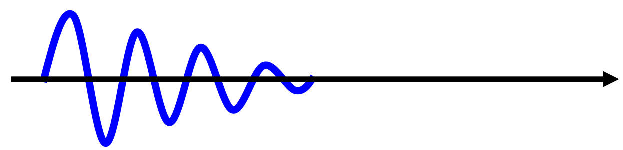
UNDERDAMPED OSCILLATOR

$$\gamma < \omega_0, \sqrt{\gamma^2 - \omega_o^2} \ \text{:imaginary}$$

$$x(t) = Be^{-\gamma t}\sin(\omega t + \theta) \qquad \omega_0^{\ 2} = \frac{k}{m} \ \& \ \gamma = \frac{c}{2m}$$

ZEROES are repetitive; strictly periodic; occur at a time period of T=2π/ω, called "period of the damped oscillator".

The number of oscillations in



a small time interval δt

$$N(\text{in } \delta t) = \frac{\delta t}{T} = \nu \delta t = \frac{\omega \delta t}{2\pi}$$

$$\nu = \frac{1}{T}; \text{ frequency}$$

In two successive periods 'T', the amplitude falls according to the following ratio:

$$\frac{B_{n+1}}{B_n} = \frac{Be^{-\gamma(t+T)}}{Be^{-\gamma t}} = e^{-\gamma T} = e^{-\varphi}$$

Logarithmic decrement factor

UNDERDAMPED OSCILLATOR

$\gamma < \omega_0, \sqrt{\gamma^2 - \omega_o^2}$:imaginary

$$x(t) = Be^{-\gamma t}\sin(\omega t + \theta)$$

Logarithmic decrement factor

In two successive periods ‘T’, the amplitude falls according to the following ratio:

$$\frac{B_2}{B_1} = \frac{Be^{-\gamma(t+T)}}{Be^{-\gamma t}}$$

$$\frac{B_2}{B_1} = e^{-\gamma T} = e^{-\varphi}$$

Question: By what amount does the amplitude diminish over a time $\delta t = NT$?

$$Now, \frac{B_{N+1}}{B_1} = e^{-\gamma NT} = e^{-N\varphi},$$

hence, when $\gamma = \frac{1}{NT}$,

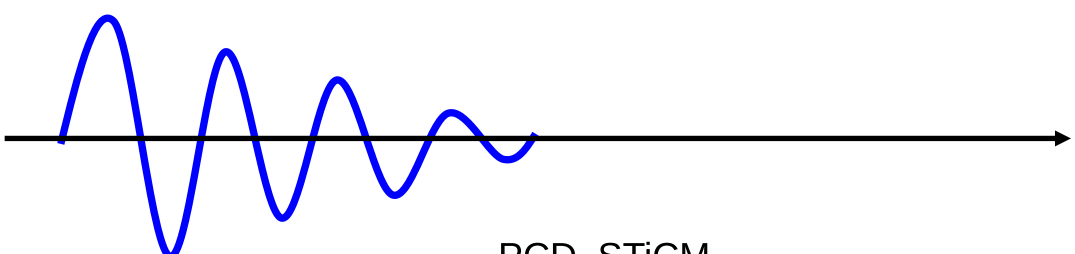
the 'amplitude decrease factor' would be $1/e$.

UNDERDAMPED OSCILLATOR

$\gamma < \omega_0, \ \sqrt{\gamma^2 - \omega_o{}^2}$ is an imaginary number

$$x(t) = Be^{-\gamma t}\sin(\omega t + \theta)$$

Unlike the 'overdamped oscillator' (no oscillations), we do have oscillations that are 'damped', not 'killed'; hence called UNDERDAMPED OSCILLATIONS



Case 3: 'CRITICAL DAMPING'

$$q_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_o^2}$$

$\gamma = \omega_0, \; q_1 = q_2 = q$: the two roots are equal

$$x(t) = Ae^{qt}$$

Can we get the 2nd linearly independent solution by considering the following simplest departure from the previous one?

$$x(t) = Bte^{-\gamma t}$$

$$x(t) = Ae^{-\gamma t} + Bte^{-\gamma t} = (A + Bt)e^{-\gamma t}$$

At $t = -\frac{A}{B}$, the system reaches the equilibium position,

and then, after the overshoot,

the next attainment of equilibrium can be

only after infinite time.

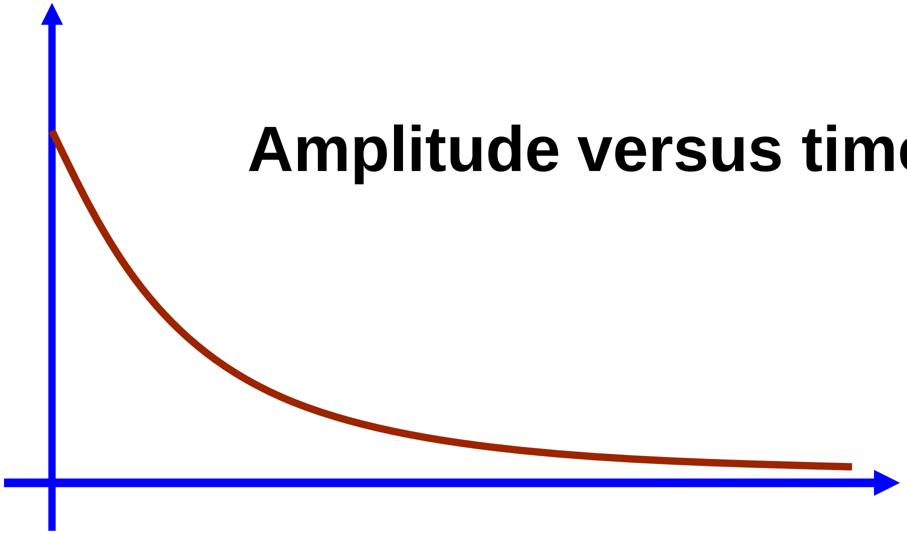
Let us recapitulate main results!

Overdamped Oscillator

When $\gamma > \omega_0$, i.e. $c^2 > 4mk$,

$\sqrt{\gamma^2 - \omega_o^{\ 2}}$ is a real number whose value is $< \gamma$,

so both q_1 and q_2 become 'real' and essentially 'negative'



Amplitude versus time

No oscillation !
No overshoot off equilibrium

Underdamped Oscillator

$$x = A_1 e^{q_1 t} + A_2 e^{q_2 t}$$

When $\gamma < \omega_0$, $\sqrt{\gamma^2 - \omega_o^{\ 2}}$ is an imaginary number; $c^2 < 4mk$,

$$q_1 = -\gamma + i\omega; \quad q_2 = -\gamma - i\omega; \quad \textit{where } \omega = \sqrt{\omega_0^{\ 2} - \gamma^2}$$

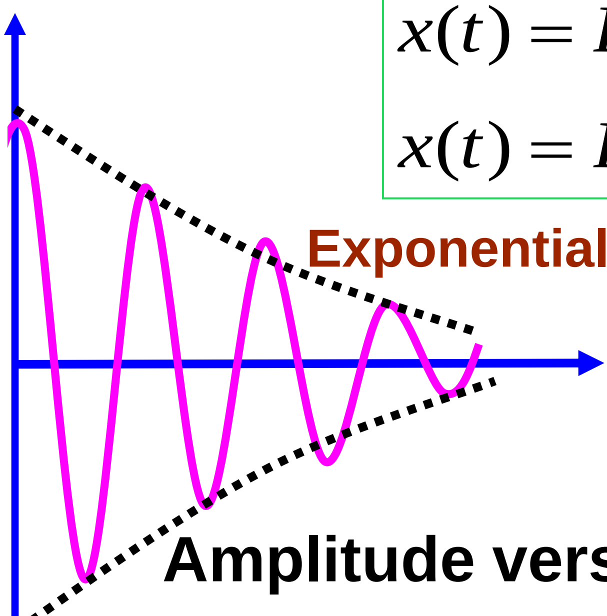
$i.e., \omega < \omega_0$ by an amount determined by γ

$$\gamma = \frac{c}{2m}$$

$$x = e^{-\gamma t}\left\{(A_1 + A_2)\cos(\omega t) + i(A_1 - A_2)\sin(\omega t)\right\}$$

$$x(t) = Be^{-\gamma t}\left\{\sin\theta\cos(\omega t) + \cos\theta\sin(\omega t)\right\}$$

$$x(t) = Be^{-\gamma t}\sin(\omega t + \theta)$$



‘zero/equilibrium crossings’ ***do*** occur

Oscillations damped, ***not killed !***

Amplitude diminishes more rapidly for larger values of *c*

Critically damped oscillator

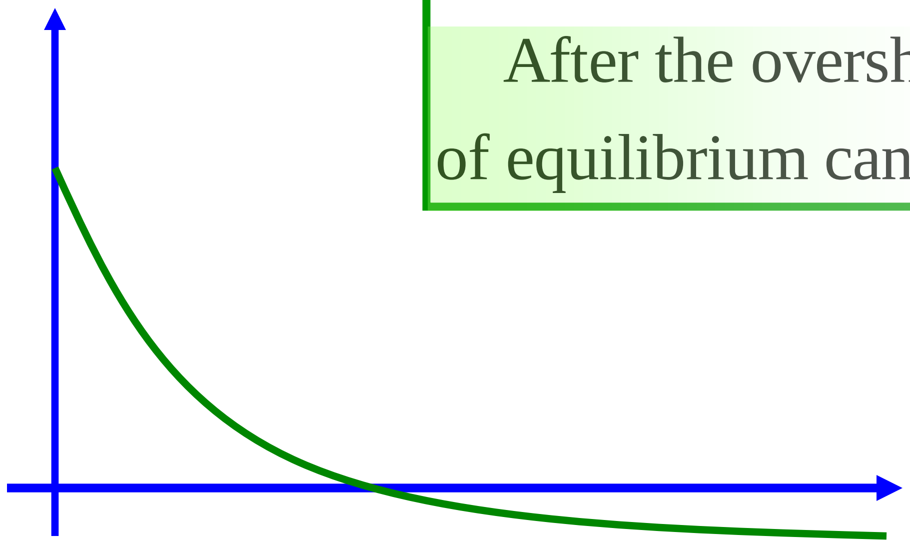
$$\gamma = \omega_0, \; c^2 = 4mk, \; q_1 = q_2 = q \text{ : the two roots are equal}$$

$$x(t) = Ae^{-\gamma t} + Bte^{-\gamma t} = (A + Bt)e^{-\gamma t}$$

The equilibrium position x=0 is reached

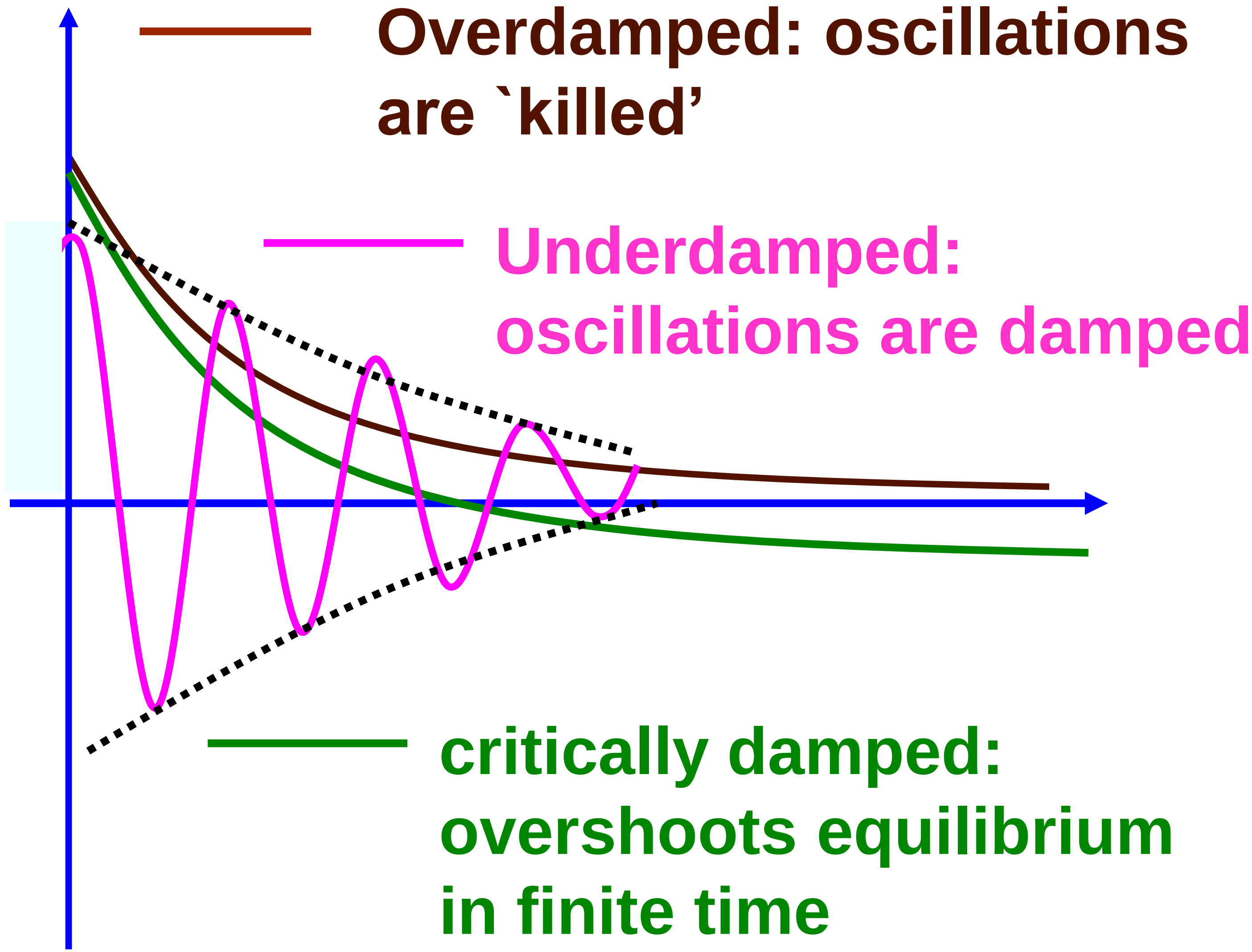
in 'finite' time interval, $t = -\frac{A}{B}$.

After the overshoot, the next attainment of equilibrium can be only after 'infinite' time.



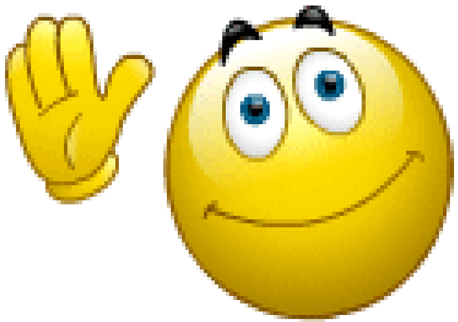
Amplitude versus time

Amplitude versus time: all the three cases



We will take a break ……

…… ANY QUESTIONS ?



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Next:

Forced oscillations

Restoring force, damping force

and driving force…..

RESONANCES….. Waves……

STiCM

Select / Special Topics in Classical Mechanics

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STiCM Lecture 09: Unit 2 Oscillators, Resonances, Waves

Forced oscillations

Restoring force, damping force and driving force

"**The Hand That Rocks The Cradle,**
Is The Hand That Rules The World"

-William Ross Wallace

This poem was first published in 1865 under the title "**What Rules The World**".

Forced oscillations

Restoring force, damping force and driving force

$$F = m\ddot{x} = -kx - c\dot{x} + F_{dr}$$

i.e.,

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_{dr}}{m}$$

For a simple pendulum with damping ,

$$\ddot{\theta} + \frac{c}{ml}\dot{\theta} + \frac{g}{l}\theta = \frac{F_{dr}}{ml}$$

For an LCR oscillator,

$$\ddot{Q} + \frac{R}{L}\dot{Q} + \frac{1}{LC}Q = \frac{V_{dr}}{L}$$

or, $L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = V_{dr}$

$$F = m\ddot{x} = -kx - c\dot{x} + F_{dr} \text{ or } \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F_{dr}}{m}$$

Actual form of the solution depends on the functional form of F_{dr}

Let $F_{dr} = F_0 e^{i(\Omega t + \theta)}$, a periodic force, with frequency Ω

θ is a phase angle - depends on 'when' we 'start' the driving force

$$\omega_0^{\ 2} = \frac{k}{m} \ \& \ \gamma = \frac{c}{2m}$$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i(\Omega t + \theta)}$$

Special case: No damping

$$\ddot{x} + \omega_o^{\ 2} x = \left[\frac{F_o}{m} e^{i(\Omega t + \theta)}\right].$$

Complex amplitude which includes time-independent phase $e^{i\theta}$

$$F_{dr} = F_o e^{i\theta} e^{i\Omega t} = \hat{F} e^{i\Omega t}$$

where $\hat{F} = F_o e^{i\theta}$

$$\ddot{x}+\omega_o{}^2 x = \left[\frac{F_o}{m} e^{i(\Omega t+\theta)}\right].$$

$$F_{dr} = \widehat{F} e^{i\Omega t} \quad \text{where } \widehat{F} = F_o e^{i\theta}$$

$x(t) = \widehat{x} e^{i\Omega t}$, (where $\widehat{x}$ includes the phase factor) is a solution of the differential equation for damped, forced vibrations

$$\dot{x} = i\Omega x,$$

$$\ddot{x} = (i\Omega)^2 x$$

The exponential form allows us to interpret the effect of differentiation with respect to time through the operator (d/dt) to be equivalent to multiplication by ($i\Omega$)

Using above relations in

$$\ddot{x}+\omega_o{}^2 x = [F_o e^{i(\Omega t+\theta)}]/m,$$

we get $(\omega_0^2 - \Omega^2)\widehat{x} = \widehat{F}/m$,

$$\widehat{x} = \frac{\widehat{F}}{\left\{m(\omega_o^2 - \Omega^2)\right\}}$$

Note!

Ω, the driving frequency becomes equal to ω_0, the natural frequency, the amplitude blows up to infinity.

General case, including damping:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\exp\ i(\Omega t + \theta)$$

Note! There are two angles to keep track of!

Try a solution $x(t) = \hat{A}e^{i\Omega t}$

with $\hat{A} = A_0 e^{i(\theta - \phi)}$

$\dot{x} = \hat{A}(i\Omega)e^{i\Omega t} = i\Omega x$

$\ddot{x} = \hat{A}(i\Omega)^2 e^{i\Omega t} = -\Omega^2 x$

θ: 'Timing' – when exactly do you start applying the driving force

ϕ: phase lag of oscillation *w.r.t.* the driving force

$$F_{dr} = F_o e^{i\theta} e^{i\Omega t} = \hat{F} e^{i\Omega t}$$

where $\hat{F} = F_o e^{i\theta}$

Substituting for $\dot{x}$ and $\ddot{x}$:

$$[-\Omega^2 + i(2\gamma\Omega) + \omega_0^2]\, x(t) = (\hat{F}/m)e^{i\Omega t}$$

$$i.e. [(\omega_0^2 - \Omega^2) + i2\gamma\Omega]\ \hat{A}\ e^{i\Omega t} = (\hat{F}/m)e^{i\Omega t}$$

General case, including damping:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\exp\ i(\Omega t + \theta)$$

$$F_{dr} = F_o e^{i\theta} e^{i\Omega t} = \widehat{F} e^{i\Omega t}$$

$$\text{where } \widehat{F} = F_o e^{i\theta}$$

$$x(t) = \widehat{A} e^{i\Omega t}$$

$$\text{with } \widehat{A} = A_0 e^{i(\theta-\phi)}$$

$$[(\omega_0^2 - \Omega^2) + i2\gamma\Omega]\, \widehat{A}\, \cancel{e^{i\Omega t}} = (\widehat{F}/m)\cancel{e^{i\Omega t}}$$

$$\widehat{A} = \frac{\left\{\widehat{F}/m\right\}}{\left\{\left(\omega_0^2 - \Omega^2\right) + i2\gamma\Omega\right\}}$$

$$A_0 e^{i(\theta-\phi)} = \frac{\left\{F_0 e^{i\theta}/m\right\}}{\left\{\left(\omega_0^2 - \Omega^2\right) + i2\gamma\Omega\right\}}$$

$$\text{as } \widehat{F} = F_0 e^{i\theta}$$

cancel

$$A_0 e^{i(\theta-\phi)} = \frac{\{F_0 e^{i\theta}/m\}}{(\omega_0^2-\Omega^2)+i2\gamma\Omega}; \quad A_0 e^{-i\phi} = \frac{\{F_0/m\}}{(\omega_0^2-\Omega^2)+i2\gamma\Omega}$$

$$e^{-i\phi} = \frac{\{F_0/(mA_0)\}}{(\omega_0^2-\Omega^2)+i2\gamma\Omega}$$

Separate now the real and imaginary parts
by multiplying both numerator and denominator
by the complex conjugate of the denominator

$$\cos\phi = \frac{\{F_0/(mA_0)\}(\omega_0^2-\Omega^2)}{(\omega_0^2-\Omega^2)^2+4\gamma^2\Omega^2} \text{ and } \sin\phi = \frac{\{F_0/(mA_0)\}2\gamma\Omega}{(\omega_0^2-\Omega^2)^2+4\gamma^2\Omega^2}$$

$$\cos\phi = \frac{\{F_0/(mA_0)\}(\omega_0^2-\Omega^2)}{(\omega_0^2-\Omega^2)^2+4\gamma^2\Omega^2} \text{ and } \sin\phi = \frac{\{F_0/(mA_0)\}2\gamma\Omega}{(\omega_0^2-\Omega^2)^2+4\gamma^2\Omega^2}$$

$$and \quad \phi = \tan^{-1}\left\{\frac{2\gamma\Omega}{\omega_0^2-\Omega^2}\right\}; \quad \tan\phi = \frac{2\gamma\Omega}{\omega_0^2-\Omega^2}$$

Squaring and adding $\sin^2\phi \;\&\; \cos^2\phi$

$$A_o(\Omega) = \frac{F_0}{m\sqrt{(\omega_0^2-\Omega^2)^2+4\gamma^2\Omega^2}}$$

Recall that our solution is:

$$x(t) = \widehat{A}e^{i\Omega t}$$

$$\text{with } \widehat{A} = A_0 e^{i(\theta-\phi)}$$

Phase factor ϕ changes markedly with the frequency Ω of the driving force.

$$x(t) = \frac{(F_0/m)}{\sqrt{(\omega_0^2-\Omega^2)^2+4\gamma^2\Omega^2}}e^{i(\Omega t+\theta-\phi)}.$$

Thus the solution for $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i(\Omega t+\theta)}$ becomes

$$x(t) = \frac{(F_0/m)}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}} e^{i(\Omega t+\theta-\phi)}.$$

Physical features of the steady state solution:

The oscillation is out of step with $F_{driving}$ through the angle ϕ.

The amplitude of the oscillation is governed by the amplitude of the driving force, modulated further by the factor $1\Big/\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}$, and also by the inertia *m*

Nature of the solution depends on γ and on the proximity of Ω to ω_0 .

Fascinating applications in mechanical, electrical and many other physical systems.

$$x(t) = \widehat{A} e^{i\Omega t}$$
$$\text{with } \widehat{A} = A_0 e^{i(\theta - \phi)}$$

$$x(t) = \frac{(F_0 / m)}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}} e^{i(\Omega t + \theta - \phi)}.$$

$$A_o(\Omega) = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2}}$$

$$A_0 = A_0(\Omega)$$

As a function of the frequency of the driving force, when will the amplitude of oscillation be a maximum?

Condition for Resonance

when is $\frac{dA_0}{d\Omega} = 0$?

$$A_o(\Omega) = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2}}$$

Two frequencies are of interest

ω_0 → Intrinsic, natural frequency.

Ω → External, under our control!

In the absence of damping, the condition that the amplitude is maximum is that $\Omega = \omega_0$ ….. but what when damping is present?

Reference: Fowles 'Analytical Mechanics', *Our notation is slightly different!*

Condition for Resonance

$$A_o(\Omega) = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2}}$$

when is $\frac{dA_0}{d\Omega} = 0$?

ω_0 → Intrinsic, natural frequency.

Ω → External, under our control!

when is $$\frac{dA_0}{d\Omega} = \frac{-\frac{1}{2}\frac{F_0}{m}\left\{2(\omega_0^2 - \Omega^2)(-2\Omega) + 8\gamma^2\Omega\right\}}{\left\{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2\right\}^{3/2}} = 0?$$

The N[r] is zero when $\Omega^2 = \omega_0^2 - 2\gamma^2$ i.e. $\Omega_r = \sqrt{\omega_0^2 - 2\gamma^2}$

$\Omega \approx \omega_0 - (\gamma^2/\omega_0) = \Omega_r$, resonance frequency

condition for resonance for a damped driven pendulum

$$\Omega_r = \sqrt{\omega_0^2 - 2\gamma^2}$$

Ω_r: resonance frequency

Recall that the frequency of the unforced (underdamped) oscillator is:

$$\omega = \sqrt{\omega_0^{\ 2} - \gamma^2}$$

$$\Omega_r = \sqrt{\omega_0^2\left(1 - \frac{2\gamma^2}{\omega_0^{\ 2}}\right)}$$

$$= \omega_0\sqrt{\left(1 - \frac{2\gamma^2}{\omega_0^{\ 2}}\right)}$$

$$\approx \omega_0\left(1 - \frac{\gamma^2}{\omega_0^{\ 2}}\right)$$

$$\approx \omega_0 - \frac{\gamma^2}{\omega_0}$$

$$\omega^2 = \omega_0^2 - \gamma^2;\ \ \omega_0^2 = \omega^2 + \gamma^2$$

$$\Omega_r = \sqrt{\left(\omega^2 + \gamma^2\right) - 2\gamma^2}$$

$$\Omega_r = \sqrt{\omega^2 - \gamma^2} = \left\{\omega^2\left(1 - \frac{\gamma^2}{\omega^2}\right)\right\}^{1/2}$$

$$= \omega\left(1 - \frac{\gamma^2}{\omega^2}\right)^{1/2}$$

$$\approx \omega\left(1 - \frac{\gamma^2}{2\omega^2}\right)$$

Amplitude at Resonance

$$A_o(\Omega) = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2}}$$

ω_0 Intrinsic, natural frequency.

Ω External, under our control*!*

$$\Omega_r = \sqrt{\omega_0^2 - 2\gamma^2}$$

Ω_r: resonance frequency

$$A_o(\Omega)_{MAXIMUM} = \frac{F_0 / m}{\sqrt{\left(\omega_0^2 - \left(\omega_0^2 - 2\gamma^2\right)\right)^2 + 4\gamma^2\left(\omega_0^2 - 2\gamma^2\right)}}$$

$$A_0(\Omega)_{MAXIMUM} = \frac{F_0 / m}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$$

i.e.

$$F_0 / m = 2\gamma A_0(\Omega)_{MAXIMUM}\sqrt{\omega_0^2 - \gamma^2}$$

Using: $$F_0/m = 2\gamma A_0(\Omega)_{MAXIMUM}\sqrt{\omega_0^{\ 2} - \gamma^2}$$

in $$A_o(\Omega) = \frac{F_0}{m\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2}}$$

we get:

$$A_o(\Omega) = \frac{2\gamma A_0(\Omega)_{MAXIMUM}\sqrt{\omega_0^{\ 2} - \gamma^2}}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2}}$$

$$A_o(\Omega) = \frac{2\gamma A_0(\Omega)_{MAXIMUM}\sqrt{\omega_0{}^2 - \gamma^2}}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + 4\gamma^2\Omega^2}}$$

Approximation

$$\omega_0^2 - \Omega^2 = (\omega_0 - \Omega)(\omega_0 + \Omega)$$

$$\approx (\omega_0 - \Omega)(2\omega_0)$$

$$\gamma << \omega_0$$

$$\Omega_r \approx \omega_0$$

$$A_o(\Omega) \approx \frac{A_0(\Omega)_{MAXIMUM}\, 2\gamma\omega_0}{\sqrt{\left\{(\omega_0 - \Omega)(2\omega_0)\right\}^2 + 4\gamma^2\omega_0{}^2}}$$

Cancelling

$2\omega_0$

in Numerator

& Denominator

$$A_o(\Omega) \approx \frac{A_0(\Omega)_{MAXIMUM}\,\gamma}{\sqrt{(\omega_0 - \Omega)^2 + \gamma^2}}$$

Thus the solution for $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i(\Omega t + \theta)}$ becomes

$$x(t) = \frac{(F_0 / m)}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}} e^{i(\Omega t + \theta - \phi)}.$$

'particular' solution

We must add the solution of the corresponding homogeneous equation (that of '<u>unforced</u>' damped oscillator) as well.

This part is a transient solution consisting of oscillations of decreasing amplitude for under-damped oscillator.

The GENERAL solution for the damped driven oscillator will be

$$x(t) = Be^{-\gamma t}\sin(\omega t + \delta) + \frac{(F_0/m)}{(\omega_0^2 - \Omega^2)} e^{i(\Omega t + \theta - \phi)}$$

Damping ignored in the steady state part, but not in the transient.

$$\frac{(F_0/m)}{\sqrt{(\omega_0^2 - \Omega^2)^2 + 4\gamma^2\Omega^2}} e^{i(\Omega t + \theta - \phi)}.$$

Why?

$$x(t) = Be^{-\gamma t}\sin(\omega t + \delta) + \frac{(F_0/m)}{(\omega_0^2 - \Omega^2)}\, e^{i(\Omega t + \theta - \phi)}$$

The three circular frequencies involved :

ω_0, the natural frequency;

ω, the frequnecy of the damped oscillator

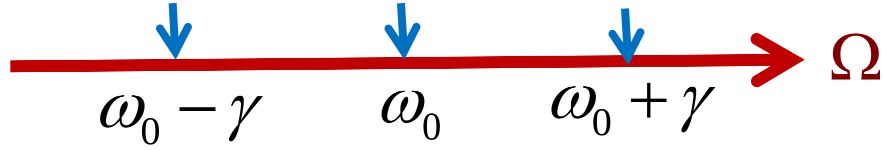
and Ω, the driving frequency

Remember! $\omega = \sqrt{\omega_0^2 - \gamma^2}$, where $\omega_o = \sqrt{k/m}$ for mass-spring oscillator,

$\omega_0 = \sqrt{\dfrac{g}{l}}$

for simple pendulum

and $\omega_o = \sqrt{\dfrac{1}{LC}}$, for LC-circuit



$$A_o(\Omega) \approx \frac{A_0(\Omega)_{MAXIMUM}\,\gamma}{\sqrt{(\omega_0 - \Omega)^2 + \gamma^2}}$$

when $\Omega = \omega_0 \pm \gamma$,

$$A_o(\Omega) = \frac{A_{0,\max}\gamma}{\sqrt{\gamma^2 + \gamma^2}} = \frac{A_{0,\max}}{\sqrt{2}}$$

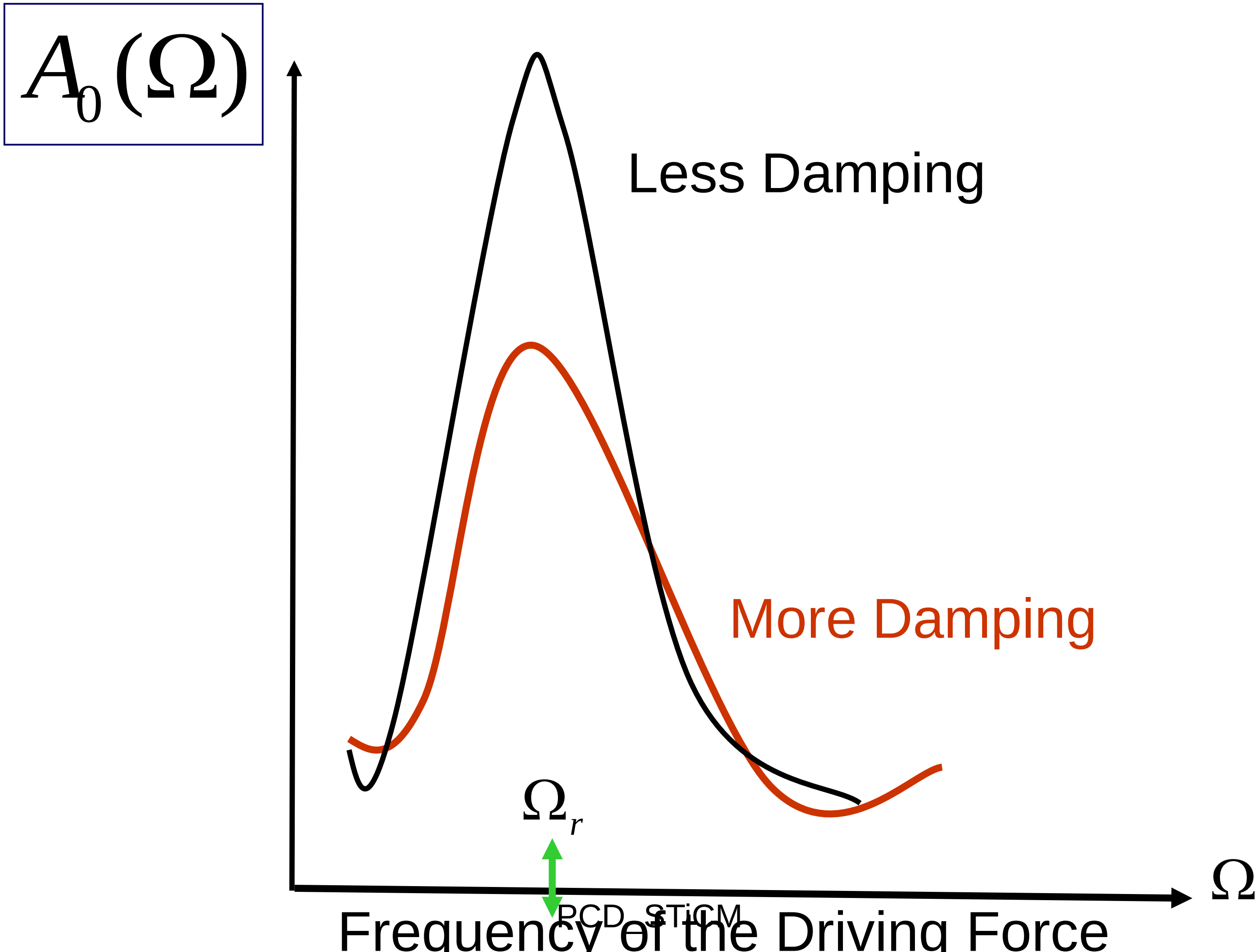
$$A_0(\Omega)^2 = \frac{1}{2} A_{\max}{}^2$$

Energy is proportional to the square of the amplitude,
and for frequencies separated by 2γ about the resonance frequency, the energy reduces by a factor of 2.

2γ "RESONANCE WIDTH"

Define: $Q = \dfrac{\omega}{2\gamma} \approx \dfrac{\omega_0}{2\gamma}$ (for the case of weak damping)

Quality Factor


$A_0(\Omega)$
Less Damping
More Damping
Ω_r
Ω
Frequency of the Driving Force

We will take a break ……

…… ANY QUESTIONS ?

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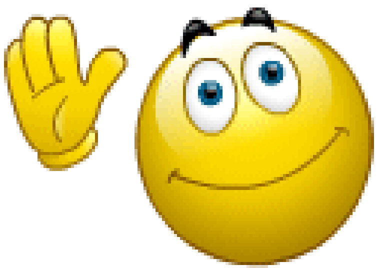

Next:

….. Waves……

STiCM

Select / Special Topics in Classical Mechanics

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STiCM Lecture 10: Unit 2 Oscillators, Resonances, Waves

The Tacoma Narrows Bridge in Washington state, was with 1.9 km length one of the largest suspended bridges built at the time. The bridge connecting the Tacoma Narrows channel collapsed in a dramatic way on Thursday November 7, 1940. Winds at about 50-70 km/hr produced an oscillation which eventually broke the construction.

Forced/Driven
Damped
Oscillator

See video of this `Disaster at Resonance' at the internet link given below!

http://www.math.harvard.edu/archive/21b_fall_03/tacoma/index.html

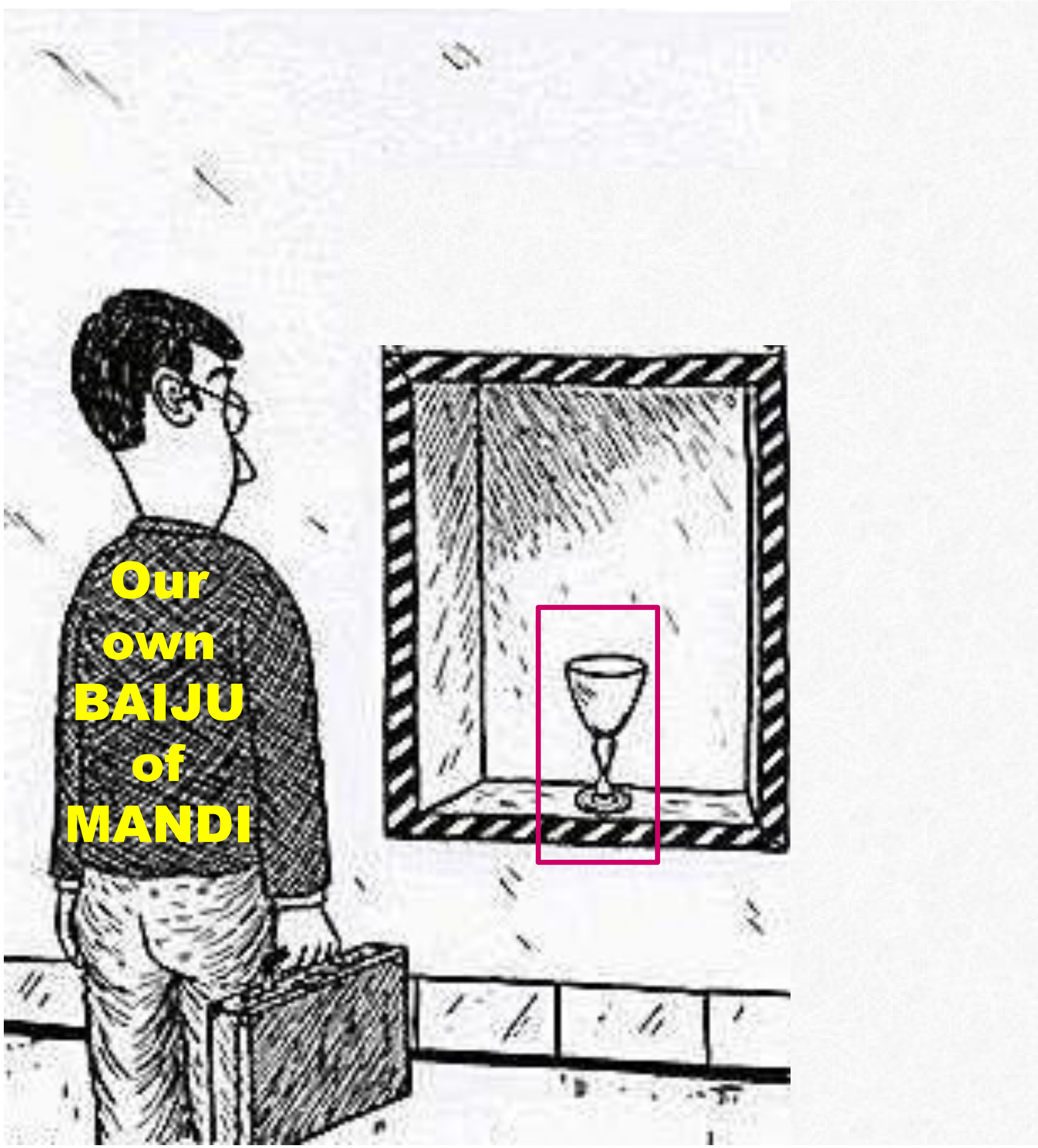
Resonances

Enrico Caruso - could shatter a crystal goblet by singing a note of just the right frequency.

Enrico Caruso
1873 - 1921

In 2005, Discovery TV Channel recruited rock singer and vocal coach Jamie Vendera to hit some crystal ware.

http://www.youtube.com/watch?v=Jy8js2FmGiY



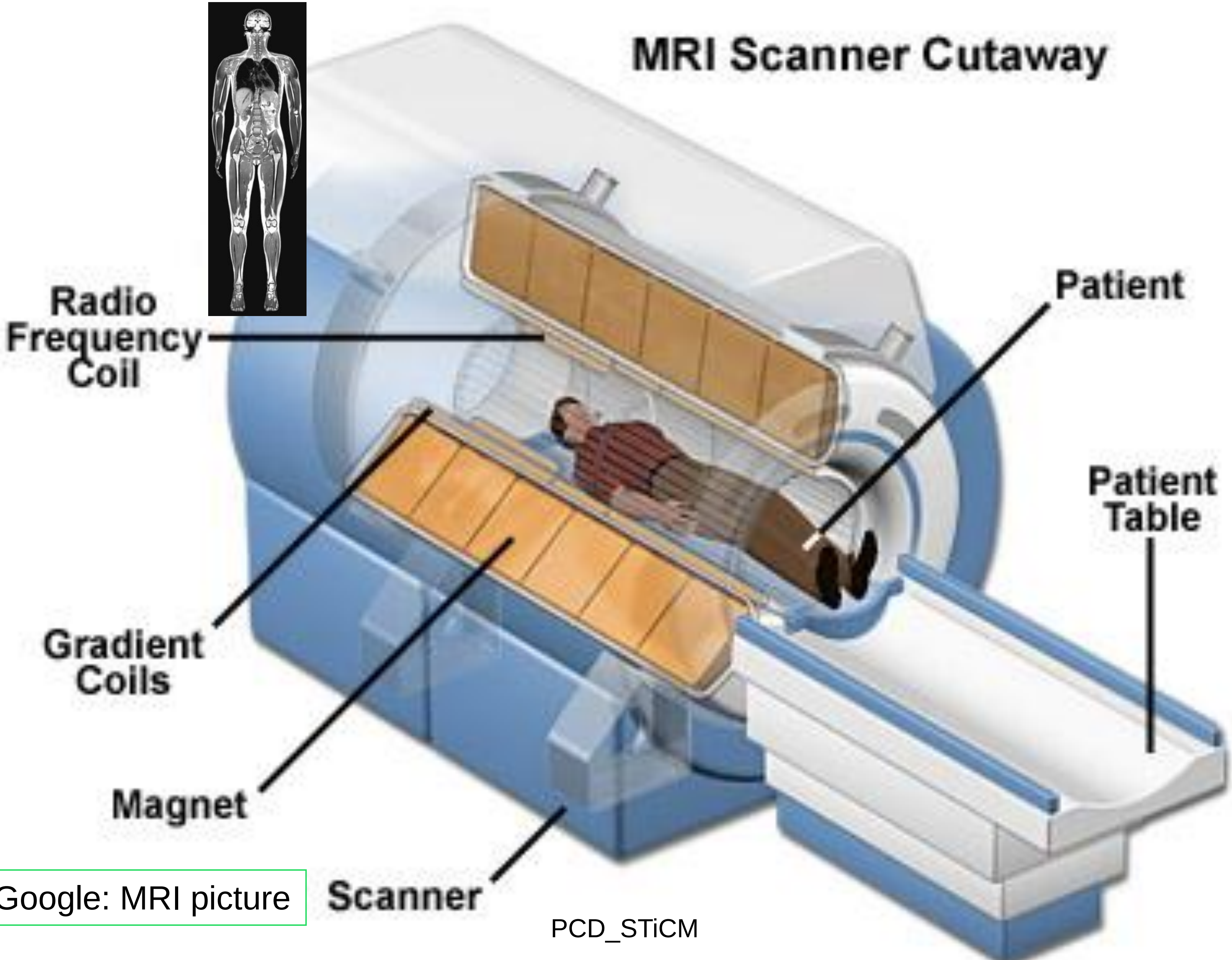
Chalo,

HAMMER

se hi kaam

chala lete

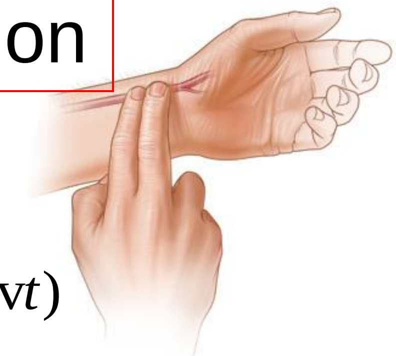
hein!



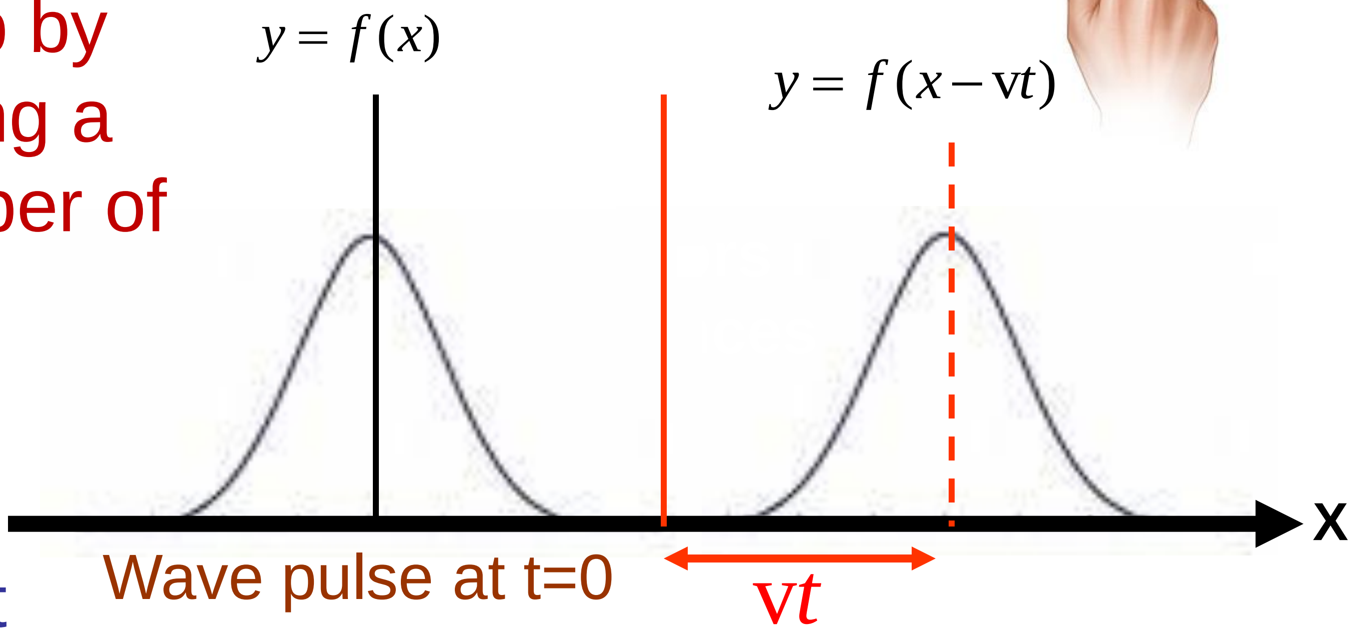
Google: MRI picture

Solutions of the oscillator problem play a fundamental, crucial role in DSP, information transmission, etc.

Wave / Pulse propagation



A wave packet, or a wave pulse, is made up by superposing a large number of sinusoidal waves.



Each component wave has a Wavelength λ

Wave pulse at t=0

Peak is at x=0

Wave pulse at t > 0

Peak is at x=vt

The component travels at its own phase velocity $v = \nu\lambda$

Shape: same

Medium: Non-dispersive

Pulse shapes ---- Fourier Analysis

Fourier :

Any periodic function can be written as a sum of simple oscillating functions

- sine and cosine functions

Jean Baptiste Joseph Fourier

March 21, 1768
May 16, 1830

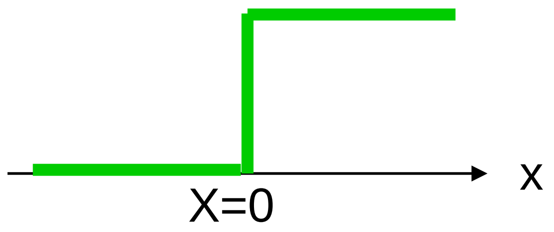
$$Plot \text{ the function: } f(x) = 2\left[H\left(\frac{x}{L}\right) - H\left(\frac{x}{L}-1\right)\right] - 1$$

when $x \in [0, 2L]$

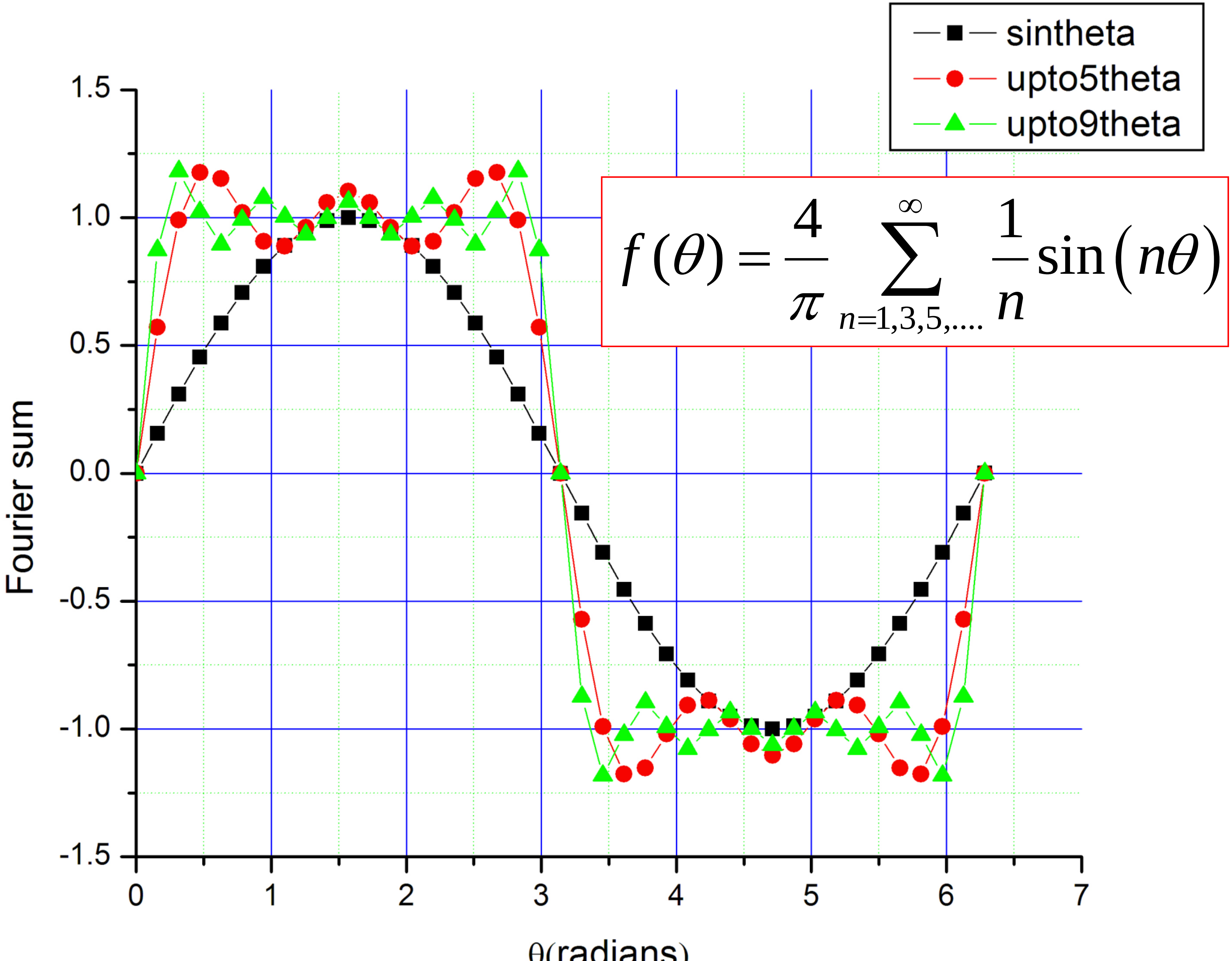
$$H(\text{x}) = 0 \text{ for x<0}$$
$$= 1 \text{ for x>0}$$

Heaviside step function

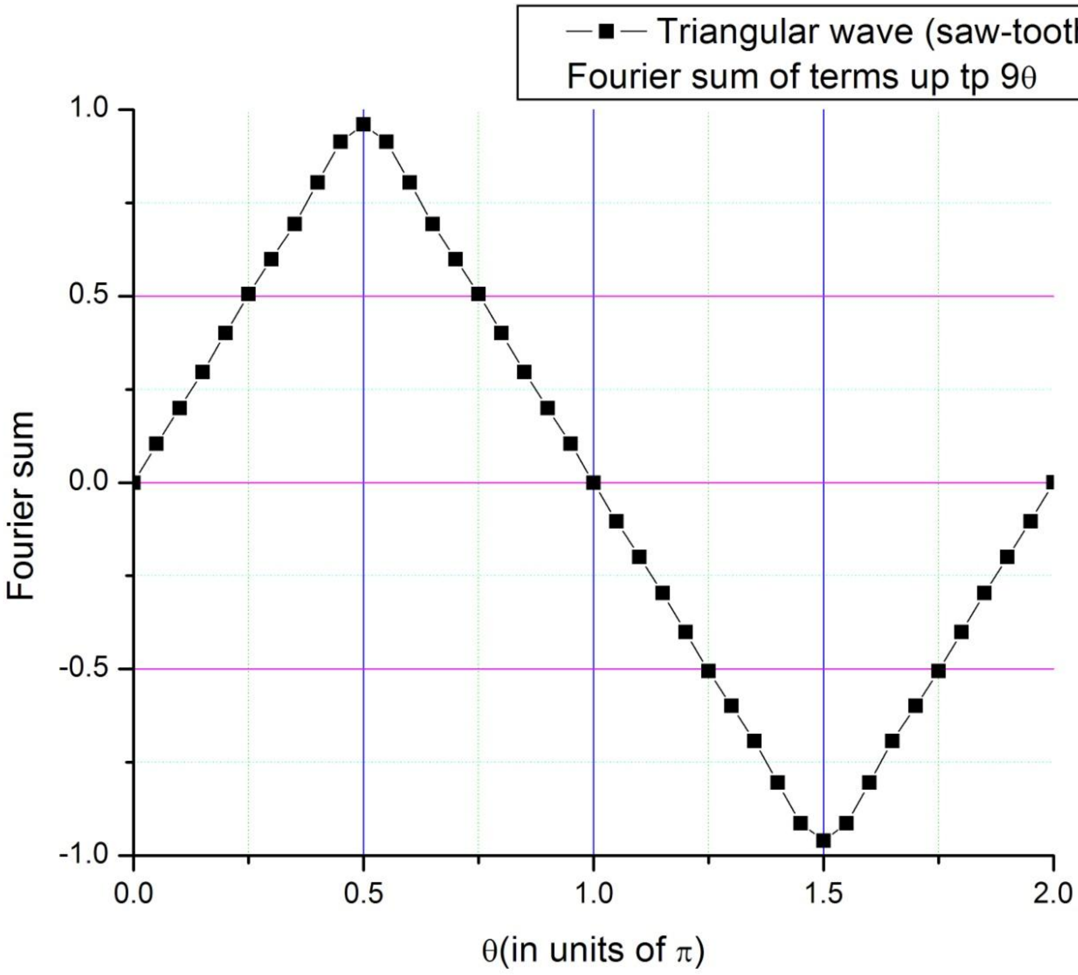
"Unit step function"



$$\text{Square Wave: } f(x) = 2\left[H\left(\frac{x}{L}\right) - H\left(\frac{x}{L}-1\right)\right] - 1$$


sintheta
upto5theta
upto9theta
$$f(\theta)=\frac{4}{\pi}\sum_{n=1,3,5,....}^{\infty}\frac{1}{n}\sin(n\theta)$$
Fourier sum
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
0
1
2
3
4
5
6
7
θ(radians)

$$f(\theta) = \frac{8}{\pi^2}\left[\sin\theta - \frac{1}{3^2}\sin(3\theta) + \frac{1}{5^2}\sin(5\theta) - \frac{1}{7^2}\sin(7\theta) + \frac{1}{9^2}\sin(9\theta) - \ldots\ldots\right]$$



Applications:

Digital Signal Processing
for example!

We worked with the function $f = f(x)$

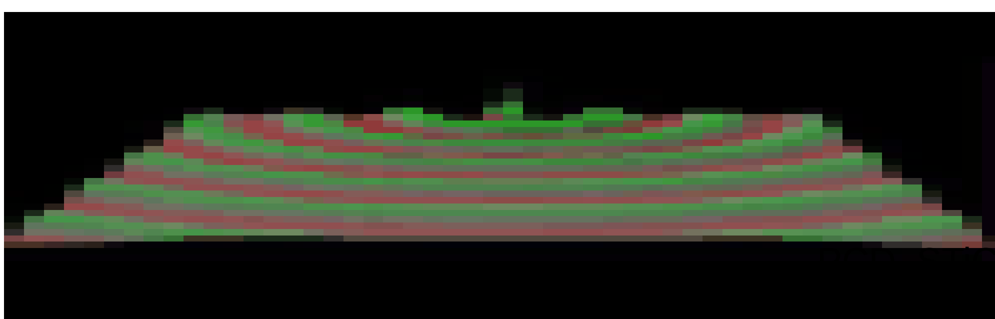
Square Wave: $f(x) = 2\left[H\left(\frac{x}{L}\right) - H\left(\frac{x}{L} - 1\right)\right] - 1$

also, we examined the saw-tooth triangular waves

In general, in wave/pulse propogations, we have function of both space and time: $f(x,t)$,

or, more generally, $f(\vec{r},t)$

often called the wavefunction $\psi(\vec{r},t)$.



$$\psi(x,t) = f(x - \mathrm{v}t)$$

If all the components of the wave-packet travel at the same speed, the ‘shape’ of the wave-packet propagates without distortion.

This is the property of a non-dispersive medium.

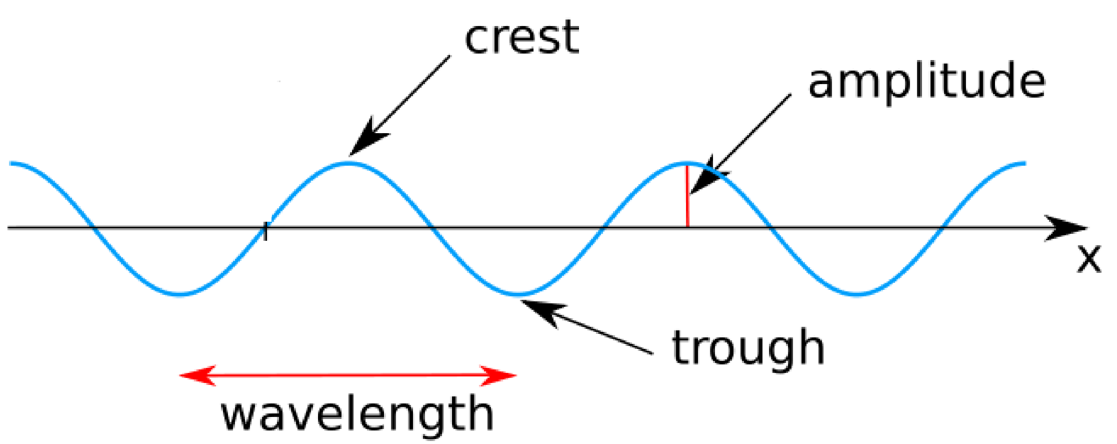
In a dispersive medium, the wave packet ‘spreads’.

$$\psi(z,t) = A\cos\omega\left\{t - \frac{z}{v_\phi}\right\} = A\cos(\omega t - kz) \qquad where\ \ k = \frac{\omega}{v_\phi}$$

phase velocity $v_\phi = \frac{\omega}{k} = \frac{2\pi\nu}{k} = \lambda\nu = \frac{\lambda}{T}$

Note:
At fixed *z*, this represents a harmonic oscillation in time.

At fixed *t*, this represents a harmonic oscillation in space.



Important parameters: frequency, period, wavelength,

amplitude, phase

‘phase’

The wavefunction $\psi(z,t) = A\cos(\omega t - kz)$

where $(\omega t - kz) = \phi(z,t)$, the phase function

At a given *z*, the phase varies linearly with time

At given *t*, the phase varies linearly with the space coordinate

In a medium, surface of constant phase is given by:

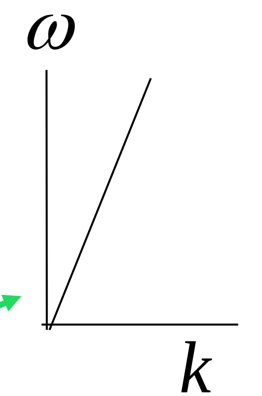
$$0 = d\varphi = \omega dt - kdz$$

$$\frac{dz}{dt} = \frac{\omega}{k} = \mathrm{v}_{\varphi}, \text{ phase velocity.}$$

‘phase velocity’ is the speed at which a wave-front defined by a surface at a certain fixed phase (e.g. a crest) advances with time.

Phase velocity $v_\phi = \frac{\omega}{k}$ for a nondispersive group of waves.

NON-DISPERSIVEWAVES: $\frac{\omega}{k}$ is constant.



In general, for dispersive waves,

v_ϕ has a much more

complicated dependence on λ (i.e. k).

ω is a function of k, given as $\omega(k)$, $v_\phi = v_\phi(k)$,

the functional form is different for different systems

Actually, it is the MEDIUM that is non-dispersive.

Properties of the MEDIUM are central to the phenomenology of NON-DISPERSIVE WAVES.

Superposition :AMPLITUDE-MODULATED TRAVELING WAVE

Superposition:

$$\psi(z,t) = A\cos(\omega_1 t - k_1 z) + A\cos(\omega_2 t - k_2 z)$$

cos A + cos B = 2 cos ½(A + B) cos ½(A - B)

cos(A + B) = cos A cos B - sin A sin B

Then, we get

$$\psi(z,t) = A_{\mathrm{mod}}(z,t)\cos(\omega_{ave} t - k_{ave} z)$$

where $A_{\mathrm{mod}}(z,t) = 2A\cos(\omega_{\mathrm{mod}} t - k_{\mathrm{mod}} z)$

$$\omega_{\mathrm{mod}} = \frac{1}{2}(\omega_1 - \omega_2); \quad k_{\mathrm{mod}} = \frac{1}{2}(k_1 - k_2)$$

also, $\omega_{ave} = \frac{1}{2}(\omega_1 + \omega_2); \quad k_{ave} = \frac{1}{2}(k_1 + k_2)$

Ref: Berkeley/ Vol.3/ Page270

$$\psi(z,t) = A_{\text{mod}}(z,t)\cos(\omega_{ave}t - k_{ave}z)$$

At what speed does the modulation propagate?

where $A_{\text{mod}}(z,t) = 2A\cos(\omega_{\text{mod}}t - k_{\text{mod}}z)$

$$\omega_{\text{mod}} = \frac{1}{2}(\omega_1 - \omega_2);\; k_{\text{mod}} = \frac{1}{2}(k_1 - k_2)$$

also, $\omega_{ave} = \frac{1}{2}(\omega_1 + \omega_2);\quad k_{ave} = \frac{1}{2}(k_1 + k_2)$

To follow a given modulation wave crest of the modulation amplitude $A_{\text{mod}}(z,t)$, we need to maintain a constant value of $(\omega_{\text{mod}}t - k_{\text{mod}}z)$

i.e., in time dt, z must increase by dz in such a way that

$$d(\omega_{\text{mod}}t - k_{\text{mod}}z) = (\omega_{\text{mod}}dt - k_{\text{mod}}dz) = 0$$

$$\psi(z,t) = A_{\text{mod}}(z,t)\cos(\omega_{ave}t - k_{ave}z)$$

At what speed does the modulation propagate?

In time dt, z must increase by dz in such a way that

$$d(\omega_{\text{mod}}t - k_{\text{mod}}z) = (\omega_{\text{mod}}dt - k_{\text{mod}}dz) = 0$$

To satisfy this, the modulation must propogate at:

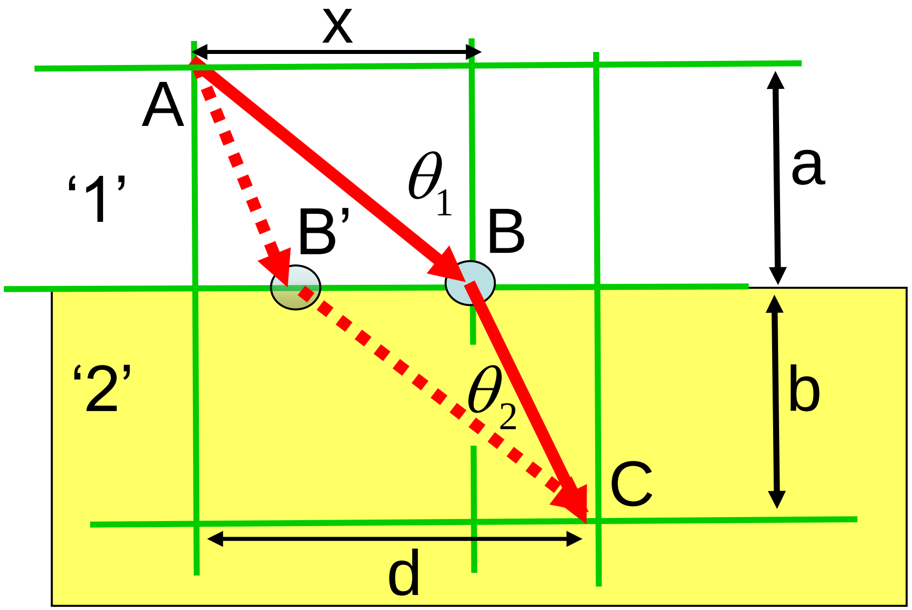
$$\frac{dz}{dt} = \text{v}_{\text{mod}} = \frac{\omega_{\text{mod}}}{k_{\text{mod}}} = \frac{\omega_1 - \omega_2}{k_1 - k_2}$$

$$= \frac{\delta\omega}{\delta k} \approx \frac{d\omega}{dk} = \text{v}_g = \text{'group velocity'}$$

If all the components of the wave-packet travel at the same speed, the 'shape' of the wave-packet propagates without distortion.

This is the property of a non-dispersive medium.

In a dispersive medium, the wave packet 'spreads'.



Refraction

Why does the light ray go along the path A → B → C, and not along A → B' → C

Time taken for light to travel the path A → B → C:

$$t = \frac{\left(a^2 + x^2\right)^{1/2}}{v_1} + \frac{\left(b^2 + (d-x)^2\right)^{1/2}}{v_2}$$

Zero (Fermat's principle)

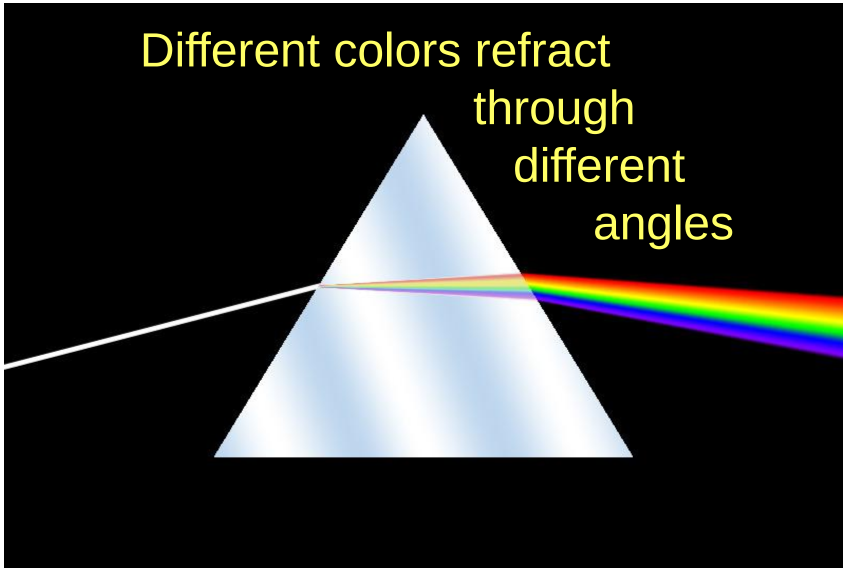
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = n_{\text{Refractive Index}}$$

Refractive index, n:

Ratio of phase velocity of light in vacuum to that in the medium

$$n = \frac{c}{v_\varphi} = \frac{\nu\lambda_{\text{vac}}}{\nu\lambda_\varphi} = \frac{\lambda_{\text{vac}}}{\lambda_\varphi}$$

$$n_r = n_r(\omega)$$



Red
Normal dispersion
Blue

Refractive Index depends on FREQUENCY in a dispersive medium

$$\omega = 2\pi\nu = \frac{c}{\lambda}2\pi = ck$$

ω vs. k graph:

constant slope, speed of light

$$\boxed{n} = \frac{\mathrm{v}_{\text{vacuum}}}{\mathrm{v}_{\text{medium}}} = \boxed{\frac{\mathrm{c}}{\mathrm{v}_{\text{medium}}}} = \frac{\cancel{\nu}\lambda_{vacuum}}{\cancel{\nu}\lambda_{medium}} = \frac{k_{medium}}{k_{vacuum}}$$

In the medium:

$$\omega = 2\pi\nu = 2\pi\frac{\mathrm{v}_{\text{medium}}}{\lambda_{\text{medium}}} = 2\pi\boxed{\frac{c}{n}}\frac{1}{\lambda_{\text{medium}}} = 2\pi\frac{\mathrm{c}}{n\lambda_{\text{medium}}}$$

$$\omega = \frac{c}{n(\nu)}k_{medium}$$

Refractive Index depends on frequency in a dispersive medium

ω vs. k graph: not linear $\Leftrightarrow$ Dispersion relation

Control speed of light ! Bring it to a halt !

Jan 18, 2001

Playing stop and go with light

http://physicsworld.com/cws/article/news/2729

2001

Storage of Light in Atomic Vapor

PRL 86:5 783

D. F. Phillips, A. Fleischhauer, A. Mair, and R. L. Walsworth
Harvard-Smithsonian Center for Astrophysics, Cambridge, Massachusetts 02138

M. D. Lukin
ITAMP, Harvard-Smithsonian Center for Astrophysics, Cambridge, Massachusetts 02138
(Received 22 December 2000)

We report an experiment in which a light pulse is effectively decelerated and trapped in a vapor of Rb atoms, stored for a controlled period of time, and then released on demand. We accomplish this "storage of light" by dynamically reducing the group velocity of the light pulse to zero, so that the coherent excitation of the light is reversibly mapped into a Zeeman (spin) coherence of the Rb vapor.

REVIEWS OF MODERN PHYSICS, VOL. 77, APRIL 2005

Electromagnetically Induced Transparency: Optics in coherent media

Laser smashes light-speed record

http://physicsworld.com/cws/article/news/2810

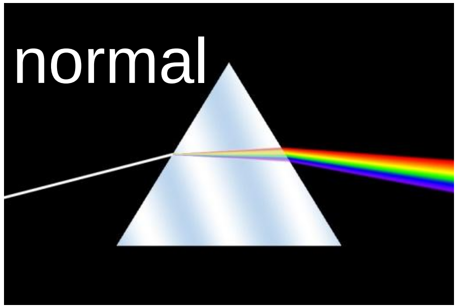
In a recent (2000) experiment at Princeton, L.J.Wang et al. managed to get a laser pulse travels at more than 300 times the speed of light *!*

L J Wang *et al*. 2000 *Nature* **406** 277

Laws of physics: intact***!***

'Normal dispersion': group velocity < phase velocity.

'Anomalous dispersion':
R.I. decreases as frequency increases; $v_{gr} > v_{ph}$.
$> c$



Red
Normal dispersion $\nu \downarrow$
Blue

Blue
Anomolous $\nu \uparrow$
Red

$$\frac{c}{v_\varphi} = n = \frac{\lambda_{vac}}{\lambda_\varphi}$$

$$n_r = n_r(\omega)$$

My heart leaps up when I behold
A rainbow in the sky:
So was it when my life began;
So is it now I am a man;
So be it when I shall grow old,
Or let me die!…

- *William Wordsworth*

R.I. of water for red is ~1.331

R.I. of water for blue is ~1.343

Questions:

1. Why is the red outside and blue inside?
2. Which part of this picture is the brightest, and why?

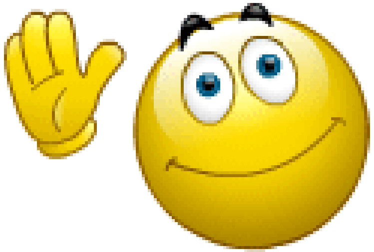
PCD_STiCM

Rainbow, seen from the 'Maid of the Mist' ride at the Niagara Falls, U.S.A., 18th July, 2009. - pcd

We Will Take a Break…

…… Any Questions ?

pcd@physics.iitm.ac.in



Next: Unit 3

Dynamical Symmetry
of the Kepler Problem

Plane polar
Cylindrical polar
Spherical polar coordinate systems