STICM

SELECT / SPECIAL TOPICS IN CLASSICAL MECHANICS

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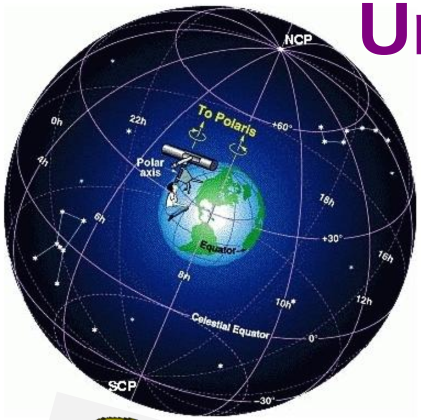
STiCM Lecture 11: Unit 3

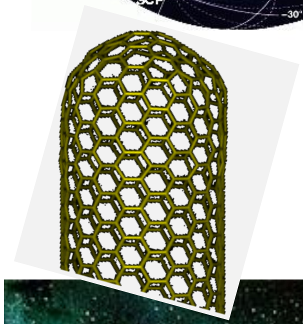
Physical Quantities – scalars, vectors….

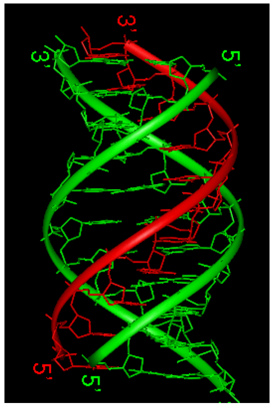
Unit 3: Polar Coordinates
Learning goals

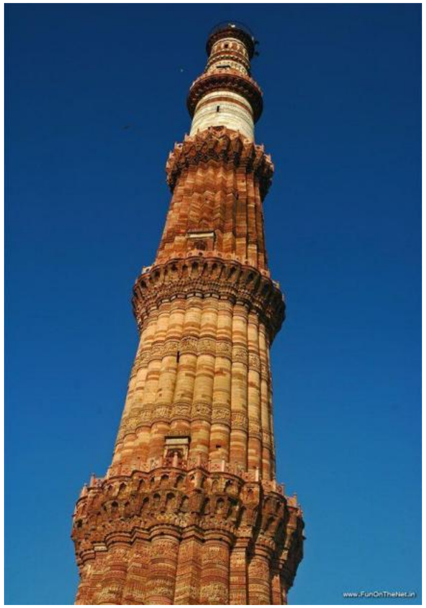


‘symmetry’

Learn to use an appropriate coordinate system to simplify analysis.

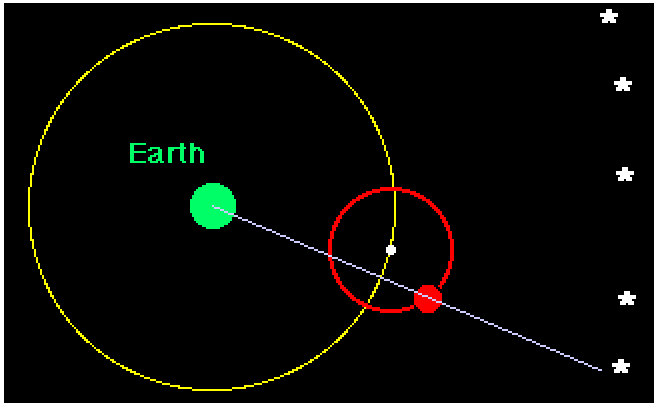






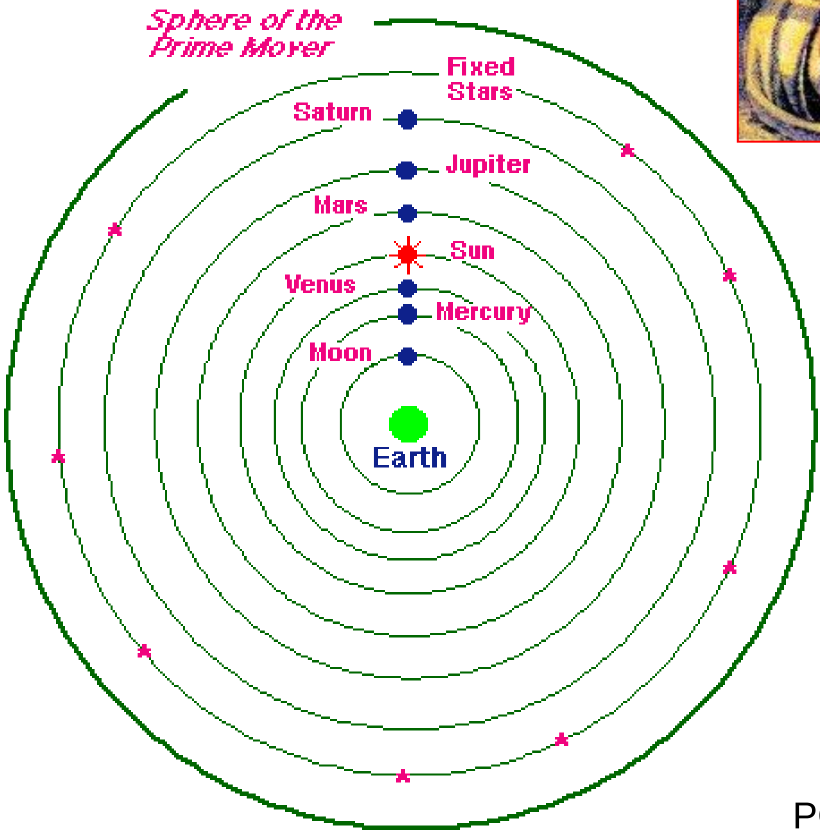
Goals: Physical quantities are tensors of various ranks.

We must examine how their ‘components’ transform under the rotation of a coordinate frame of reference.



Claudius Ptolemaeus (AD100-170) (called Ptolemy).

worked in the library of Alexandria.



Aristotle's Universe

The sun and the planets were considered to move on a small circle (called 'epicycle') whose center would move on a large circle (called 'deferent').

Contributions of Indian Astronomers to the Understanding of Heliocentric Coordinate System

Aryabhata (b. 476A.D.) - ‘ARYABHATYA’ (499 A.D.)

Bhaskara I (A.D. 600) - ‘MAHABHASKARIYA’, ‘LAGHUBHASKARIYA’, ‘ARYABHATIYA BHASHYA’

Brahmagupta (A.D. 591) - ‘BRAMA SIDDHANTA’

Vateshwa (A.D. 880) - ‘VATESHWARA SIDDHANTA’

Manjulacharya-(A.D. 932) - ‘LAGHUMANASA’
[Dealt with *Precession* of equinoxes]

Aryabhata I I (A.D. 950) - ‘MAHASIDDHANTA’

Bhaskaracharya I I (A.D. 1114) ‘SIDDHANTA SHIROMANI’ [This work contains many formulas from spherical trigonometry]……..etc.

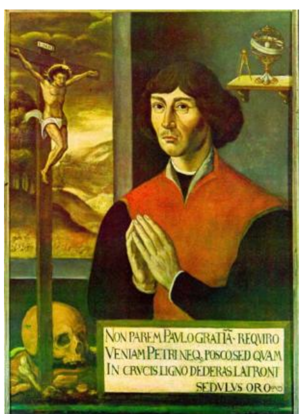
http://www.physics.iitm.ac.in/~labs/amp/kerala-astronomy.pdf

Modification of the earlier Indian planetary theory by the Kerala astronomers (c. 1500 AD) and the implied heliocentric picture of planetary motion

K. Ramasubramanian, M. D. Srinivas and M. S. Sriram

अनुलोमगतिर्नौस्थ: पश्यत्यचलं विलोमगं यद्वत् ।
अचलानि भानि तद्वत् समपश्चिमगानि लङ्कायाम् ॥ १ ॥

From Golapaada, by Aryabhata, ~500 AD



Nicolus Copurnicus
1473-1543

Just as a man in a boat moving sees the stationary objects (on either side of the river) as moving backward, so are the stationary stars seen by the people at Lanka (i.e. reference coordinate on the equator) as moving exactly toward the west.

Rene Descartes (17th century, Holland)

French philosopher, mathematician, scientist: "Father of Modern Philosophy,"

Heliocentric system vs Church's views

Despite admitting the advantages of the heliocentric coordinate system, Descartes was reluctant to promote the "certain and evident proof" in favor of the heliocentric system since it was against the will of the church.

The Trial of Galileo (for supporting Copernican model)

April 1633: Galileo is interrogated before the Inquisition.

June:
Galileo sentenced to prison for an indefinite term.

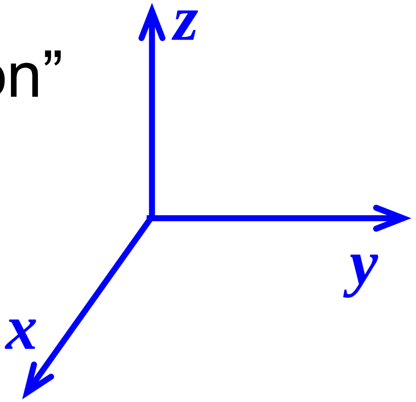
December: Galileo is allowed to return to his villa in Florence, where he lived under house-arrest.

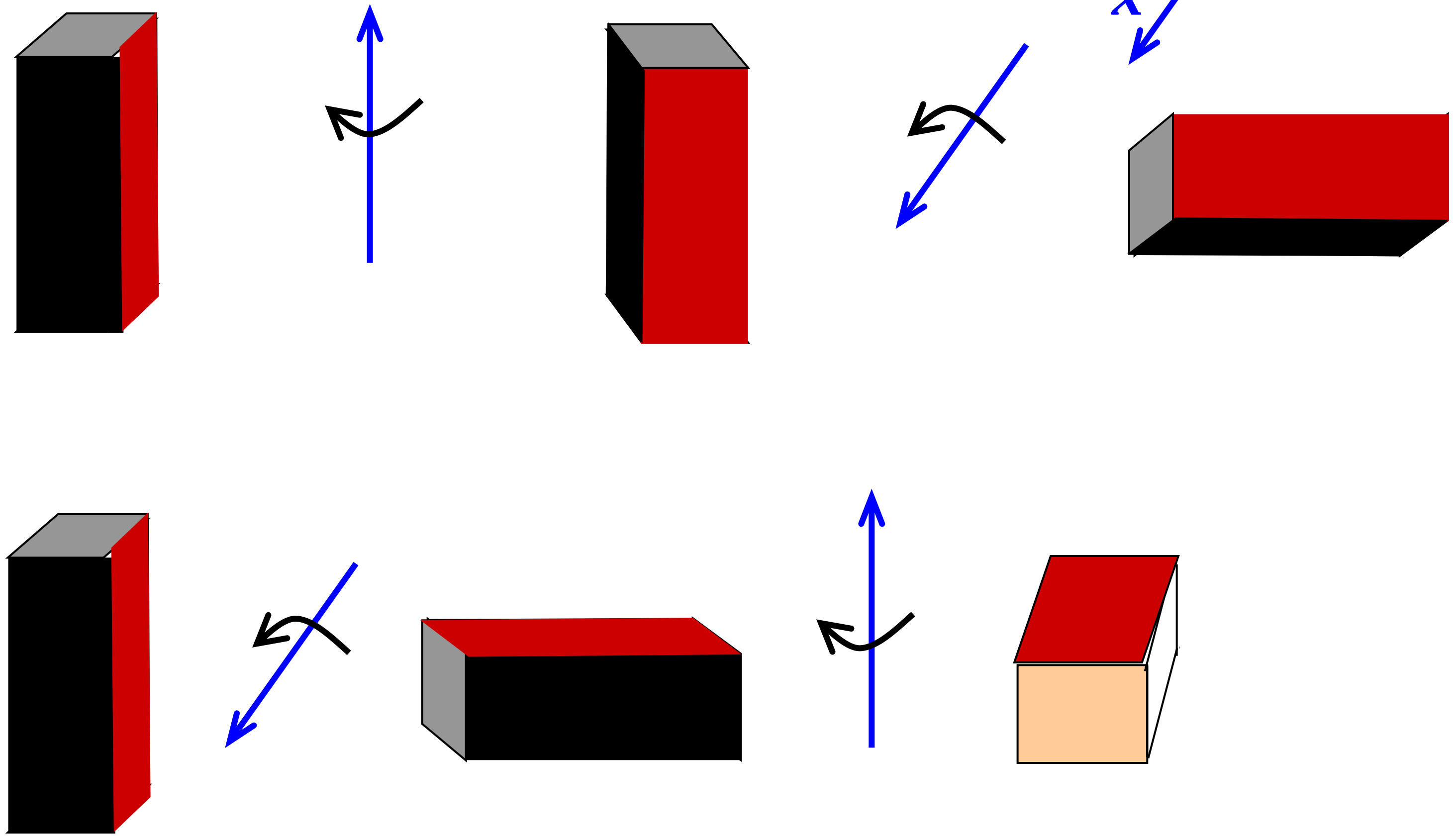
1992: Catholic Church formally admits that Galileo's views on the solar system are correct.

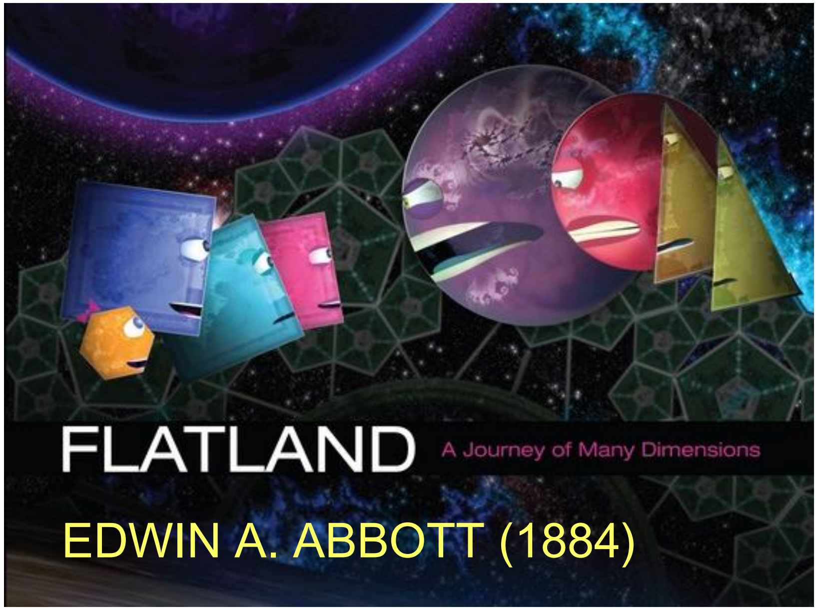
http://www.law.umkc.edu/faculty/projects/ftrials/galileo/galileochronology.html

Definition of a vector: “magnitude” and “direction”

Is rotation by 90 degrees a vector?







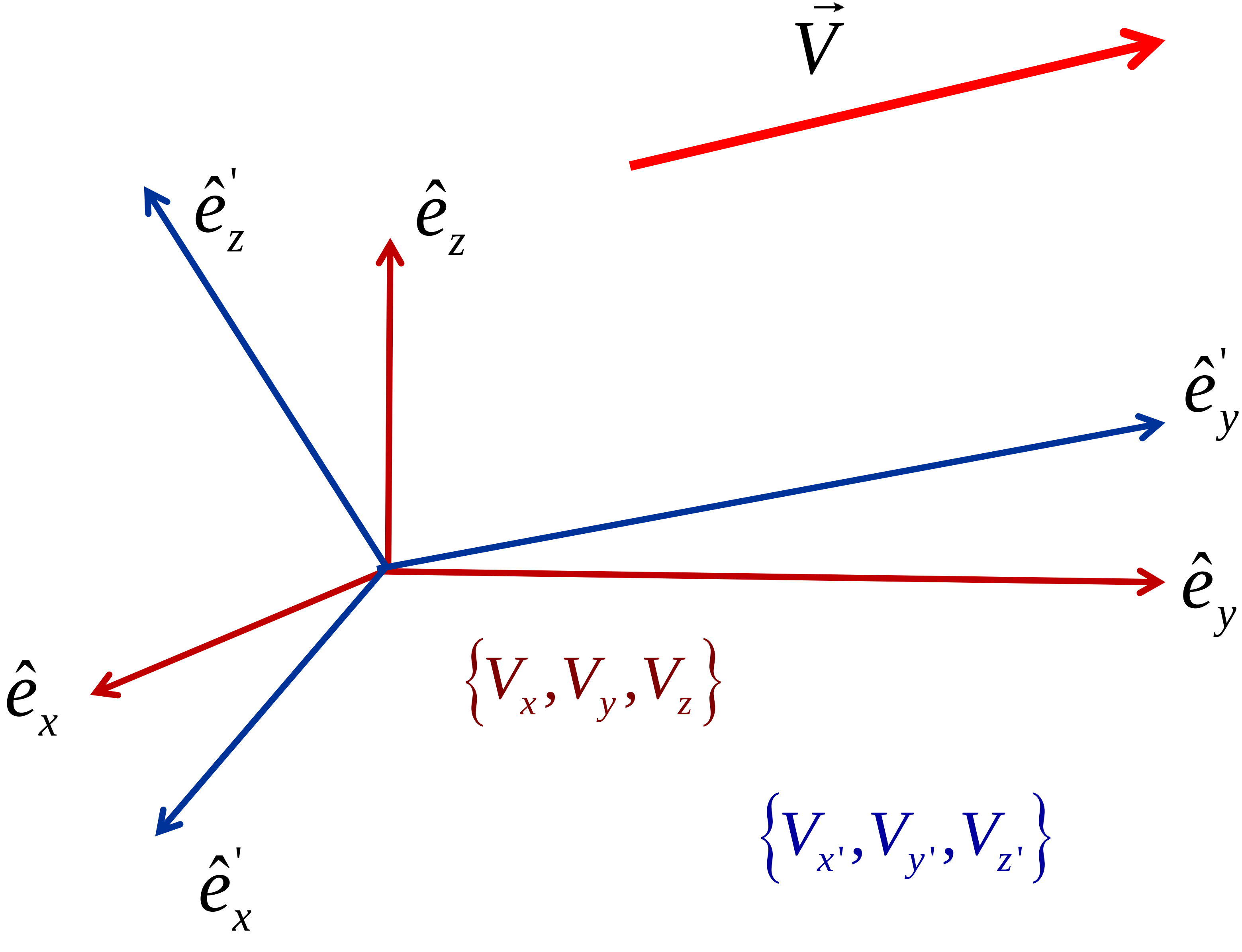
How do vectors transform under rotation of a coordinate system?

$$\vec{V} = V_x \hat{e}_x + V_y \hat{e}_y$$

Same vector can also be written as

$$\vec{V} = V_{x'} \hat{e}_{x'} + V_{y'} \hat{e}_{y'}$$

This can be generalized into three (or N) dimensions


$\vec{V}$
$\hat{e}_z'$
$\hat{e}_z$
$\hat{e}_y'$
$\hat{e}_y$
$\hat{e}_x$
$\hat{e}_x'$
$\{V_x, V_y, V_z\}$
$\{V_{x'}, V_{y'}, V_{z'}\}$

$$V_{x'} = V_x\left[\hat{e}_{x'} \cdot \hat{e}_x\right] + V_y\left[\hat{e}_{x'} \cdot \hat{e}_y\right] + V_z\left[\hat{e}_{x'} \cdot \hat{e}_z\right]$$

$$V_{y'} = V_x\left[\hat{e}_{y'} \cdot \hat{e}_x\right] + V_y\left[\hat{e}_{y'} \cdot \hat{e}_y\right] + V_z\left[\hat{e}_{y'} \cdot \hat{e}_z\right]$$

$$V_{z'} = V_x\left[\hat{e}_{z'} \cdot \hat{e}_x\right] + V_y\left[\hat{e}_{z'} \cdot \hat{e}_y\right] + V_z\left[\hat{e}_{z'} \cdot \hat{e}_z\right]$$

Compact matrix form

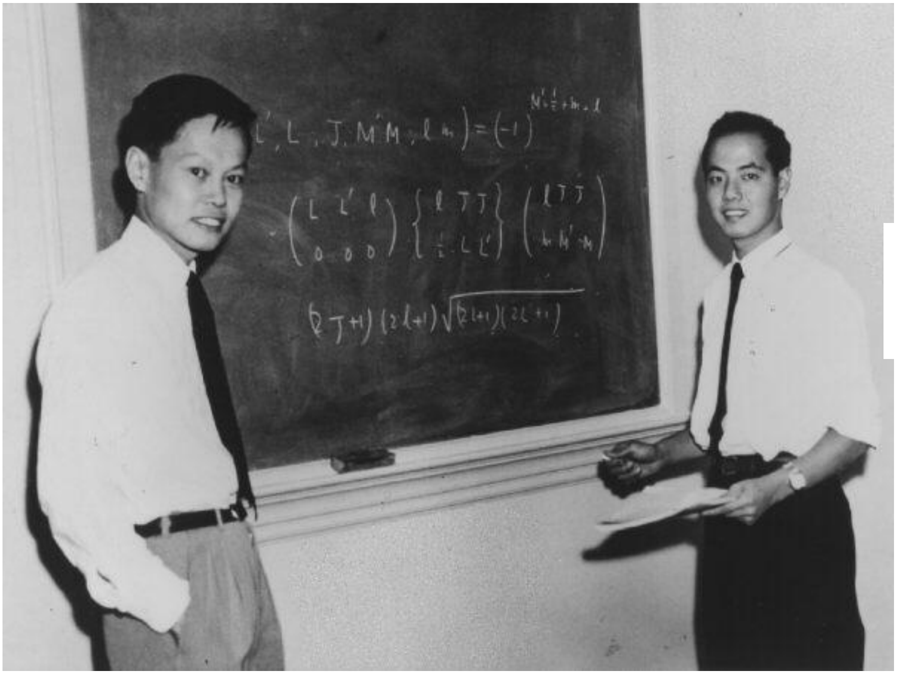
$$\begin{bmatrix} V_{x'} \\ V_{y'} \\ V_{z'} \end{bmatrix} = \begin{bmatrix} \hat{e}_{x'} \cdot \hat{e}_x & \hat{e}_{x'} \cdot \hat{e}_y & \hat{e}_{x'} \cdot \hat{e}_z \\ \hat{e}_{y'} \cdot \hat{e}_x & \hat{e}_{y'} \cdot \hat{e}_y & \hat{e}_{y'} \cdot \hat{e}_z \\ \hat{e}_{z'} \cdot \hat{e}_x & \hat{e}_{z'} \cdot \hat{e}_y & \hat{e}_{z'} \cdot \hat{e}_z \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{vmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{vmatrix} = \pm 1 \qquad \vec{r}_R = \mathbb{R}\vec{r}; \; \left|\mathbb{R}\right| = \pm 1$$

ROTATION: $|\mathbb{R}| = +1$

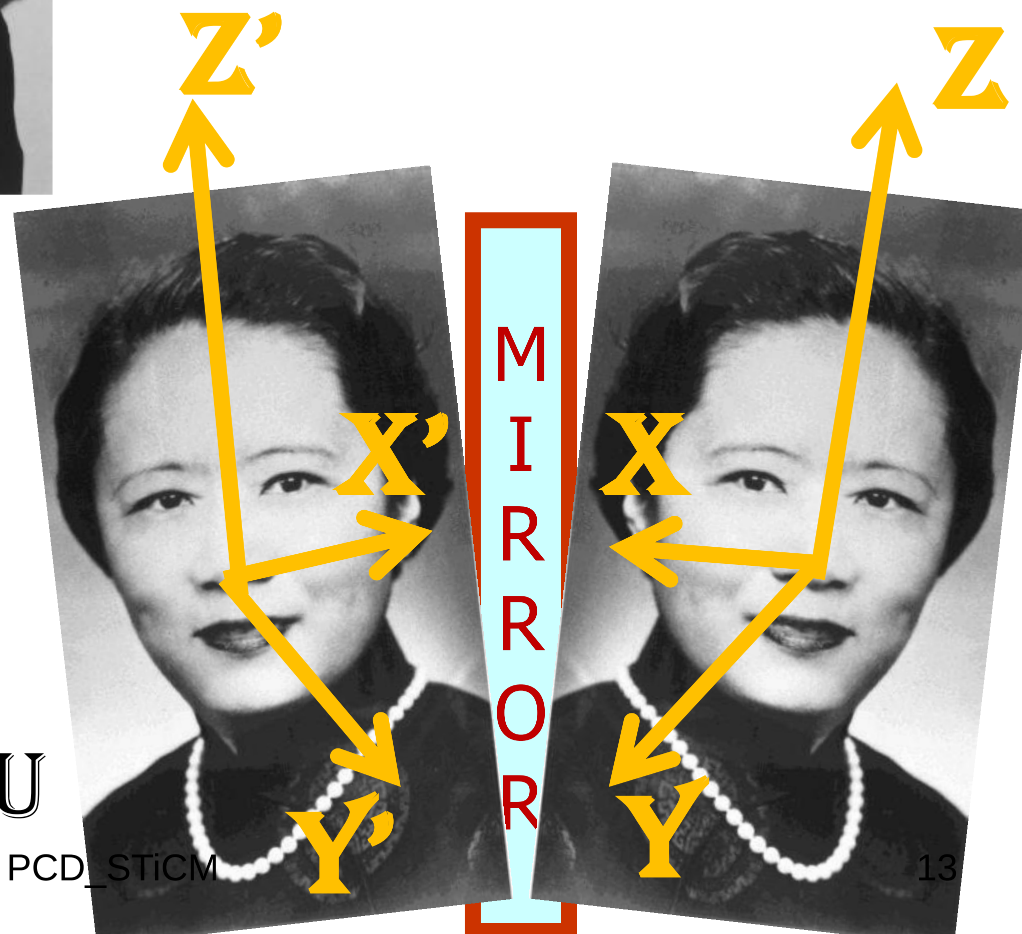
PARITY / REFLECTION INVERSION $|\mathbb{R}| = -1$



C.N.Yang and T.D.Lee

$$\begin{pmatrix} -x \\ y \\ z \end{pmatrix} \leftarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Chien-Shiung Wu



$$\begin{pmatrix} -x \\ y \\ z \end{pmatrix} \leftarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

REFLECTION

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix}$$

LEFT ⟷ RIGHT

TOP ⟷ ? BOTTOM

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$

Too much Mathematics?

"IF YOU WANT TO READ THE BOOK OF THE UNIVERSE ,

YOU MUST KNOW ITS LANGUAGE, WHICH IS MATHEMATICS".

Who said that?

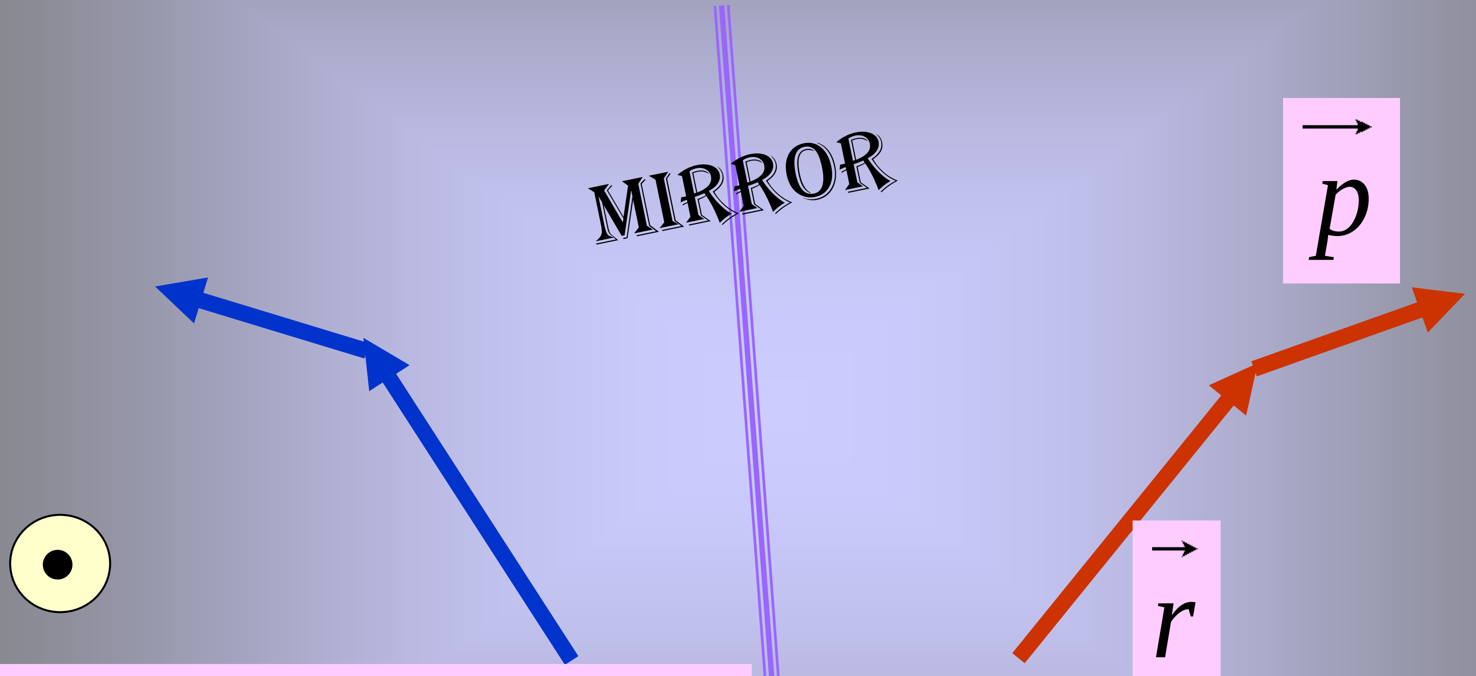
Grandpa of Engineering?

Father of experimental Physics!

$$\vec{C} = \vec{A} \times \vec{B}$$

$$\vec{l} = \vec{r} \times \vec{p}$$

angular momentum



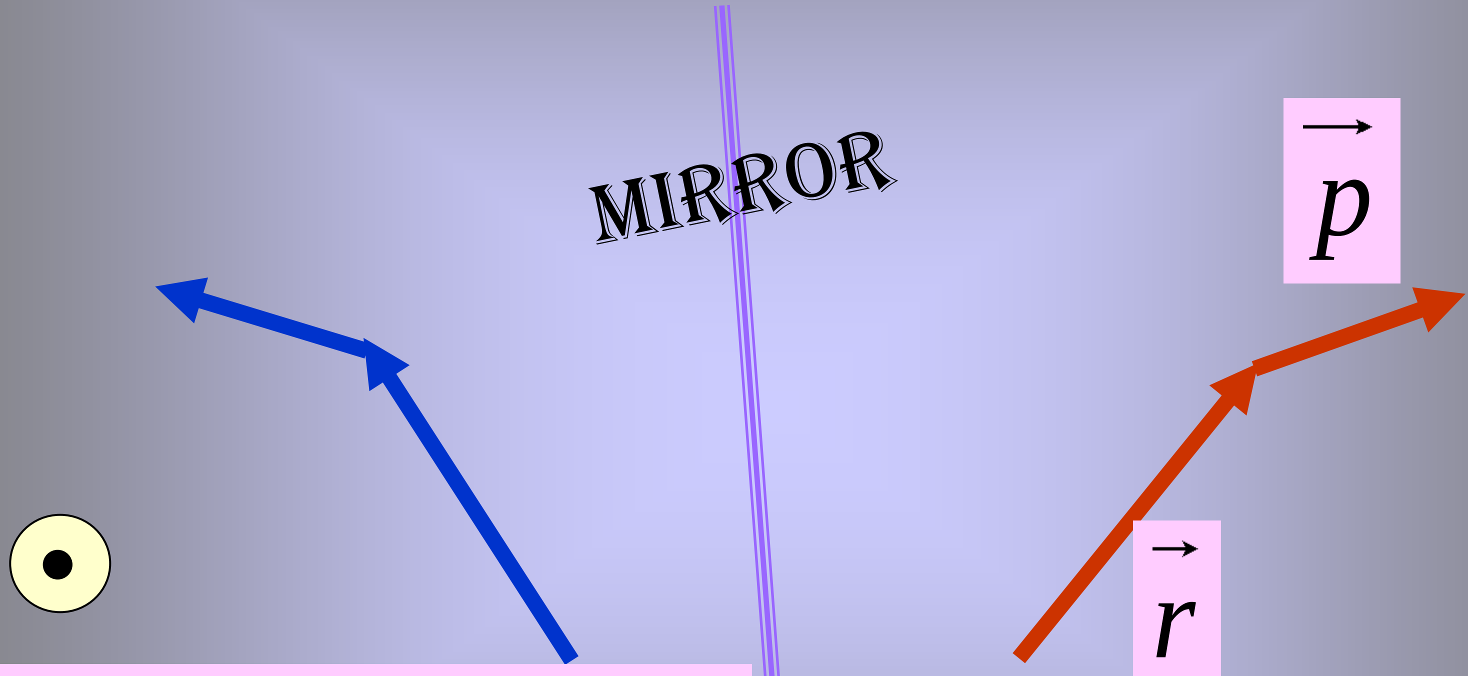
$$\vec{l}_{right-hand-cross-product} = \vec{r}_{image} \times \vec{p}_{image}$$

$$\otimes \quad \vec{l} = \vec{r} \times \vec{p}$$

$$\vec{C} = \vec{A} \times \vec{B}$$

$$\vec{l} = \vec{r} \times \vec{p}$$

angular momentum



$$\vec{l}_{right-hand-cross-product} = \vec{r}_{image} \times \vec{p}_{image}$$

$$\otimes \quad \vec{l} = \vec{r} \times \vec{p}$$

Polar vectors and pseudo- or axial-vectors

$$under\ \ inversion,$$

$$\vec{r} \to -\vec{r},$$

$$\vec{p} \to -\vec{p}$$

$$but\ \ if\ \ \vec{l} = \vec{r} \times \vec{p},$$

$$then,\ \text{under inversion,}\ \ \vec{l} \not\to -\vec{l}$$

Axial vector (pseudo vector) does not transform like a position vector under reflection.

Its components are governed by a different transformation law with respect to rotation of the coordinate system.

Examples :
Some ‘real physical quantities’

Angular Momentum Vector $\vec{r} \times \vec{p}$

Force on a charged particle moving

in an electromagnetic field

$$\vec{F} = q\left\{\vec{E} + \vec{\mathrm{v}} \times \vec{B}\right\}$$

Lorentz Force

The Lorentz force, like any other force, is a polar vector, since it includes the cross-product of a polar vector $\vec{\mathrm{v}}$ with a pseudo-vector $\vec{B}$.

Algebra of Pseudo Vectors and Examples

Dot and cross products:

Polar x Polar = Axial

Polar x Axial = Polar

Axial x Axial = Axial

Axial . Polar = Pseudo-scalar

Examples for axial (pseudo) vectors:

Torque $\vec{\tau} = \vec{r} \times \vec{f}$

Angular Momentum $\vec{L} = \vec{r} \times \vec{p}$

Magnetic field $\vec{F}_{mag.} = q\vec{v} \times \vec{B}$

Important: An axial vector can never be equated with a polar vector

We have learned that physical quantities are represented by scalars, vectors, tensors etc.

Scalars: tensors of rank zero

Vectors: tensors of rank one

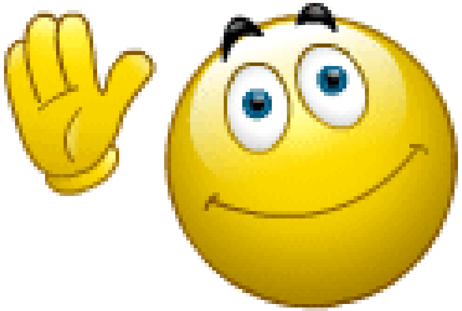
Scalars / pseudo-scalars

Vectors / pseudo-vectors

Polar vectors / Axial vectors

We will take a break…

…… Any questions ?



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Next: vectors in Polar coordinates

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STiCM Lecture 12: Unit 3

Plane Polar Coordinates

Cylindrical Polar Coordinates

Spherical Polar Coordinates

MOTORCYCLE MANIA

THE TORRES BROTHERS!

FIRST 5, THEN 7 GUYS RACE THEIR BIKES INSIDE A 16 FOOT STEEL GLOBE.

UNBELIEVABLE!

http://myspace.vtap.com/video/Motor+Cycle+Mania/CL0177433717_7cf78882_V0lLSTE1MTIxN35pbjozfnE6YnJ-Ync6V0lLSTE1MTIxNyxDTDAwNzI4MDMzMDl-aW46Nn5xOnJs

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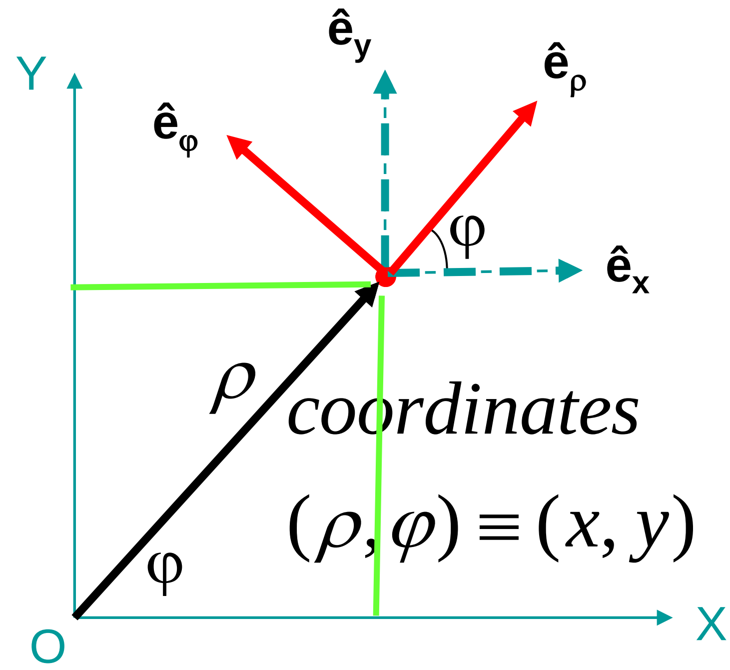
FLATLAND SPACE

$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi$$
$$-\infty < x < \infty$$
$$-\infty < y < \infty$$

$$\rho = +\sqrt{x^2 + y^2}$$
$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\rho: \quad 0 \le \rho < \infty$$
$$\varphi: \quad 0 \le \varphi < 2\pi$$



$$\hat{e}_\rho = \cos\varphi\,\hat{e}_x + \sin\varphi\,\hat{e}_y$$
$$\hat{e}_\varphi = -\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y$$

$$\hat{e}_x = \cos\varphi\,\hat{e}_\rho - \sin\varphi\,\hat{e}_\varphi$$
$$\hat{e}_y = \sin\varphi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\varphi$$

Position vector

$$\vec{\rho} = \rho\hat{e}_\rho$$

$$\hat{e}_\rho.\hat{e}_\rho = 1 = \hat{e}_\varphi.\hat{e}_\varphi$$
$$\hat{e}_\rho.\hat{e}_\varphi = 0$$

PLANE POLAR COORDINATE SYSTEM

$(\hat{e}_\rho, \hat{e}_\varphi)$ constitute an orthogonal pair of base vectors

Position vector

$$\vec{\rho} = \rho \hat{e}_{\rho}$$

VELOCITY?

ACCELERATION ?

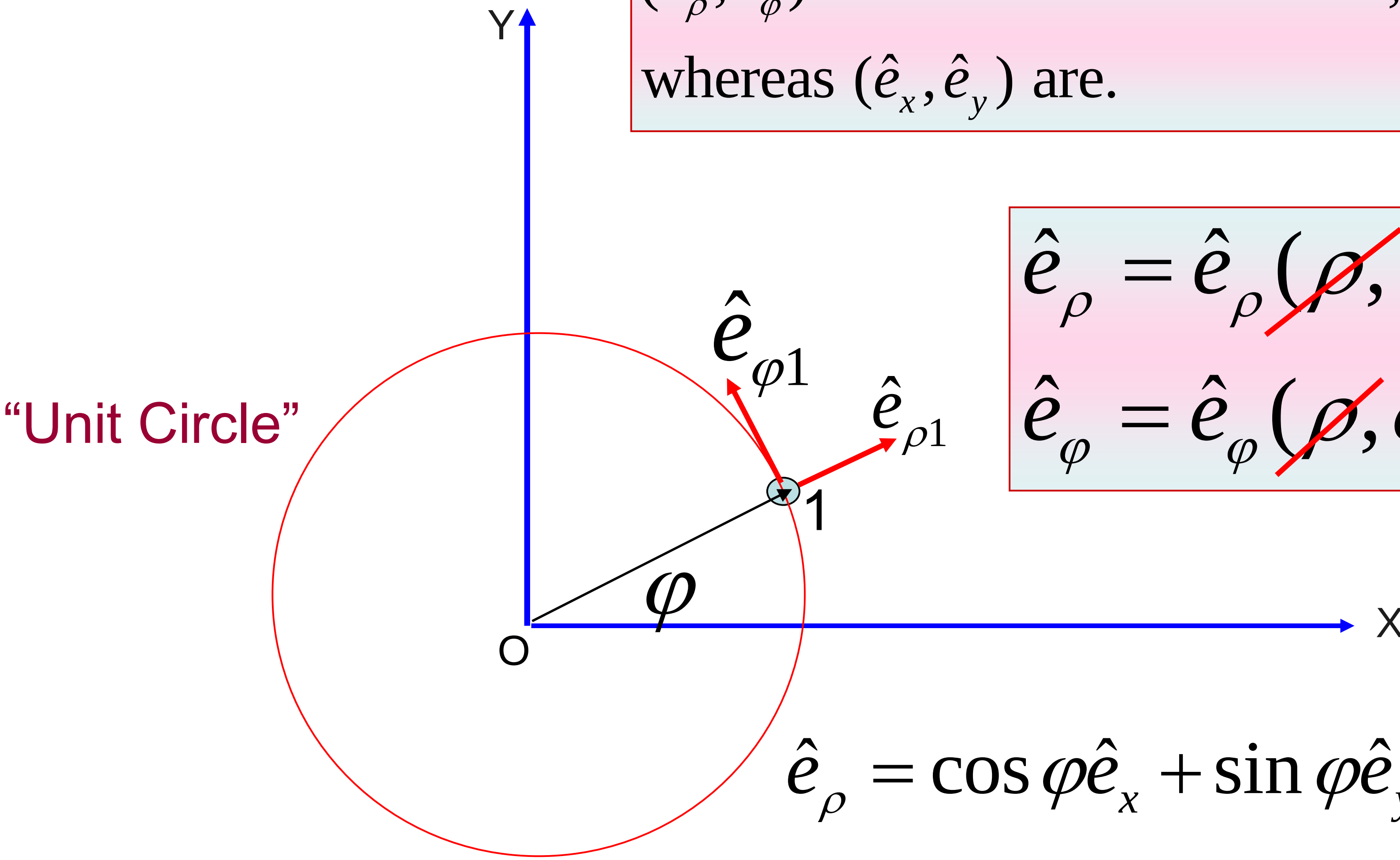
Note that $(\hat{e}_{\rho}, \hat{e}_{\varphi})$ are not constant vectors.

$$\frac{d}{dt}\left[\text{ Product of two functions }\right]$$

To get acceleration, we have to do that twice!

$$\frac{d}{dt}\frac{d}{dt}$$

$(\hat{e}_\rho, \hat{e}_\varphi)$ are not constant vectors, whereas $(\hat{e}_x, \hat{e}_y)$ are.



$$\hat{e}_\rho = \hat{e}_\rho(\cancel{\rho}, \varphi)$$

$$\hat{e}_\varphi = \hat{e}_\varphi(\cancel{\rho}, \varphi)$$

$$\hat{e}_\rho = \cos\varphi \hat{e}_x + \sin\varphi \hat{e}_y$$

$$\hat{e}_\varphi = -\sin\varphi \hat{e}_x + \cos\varphi \hat{e}_y$$

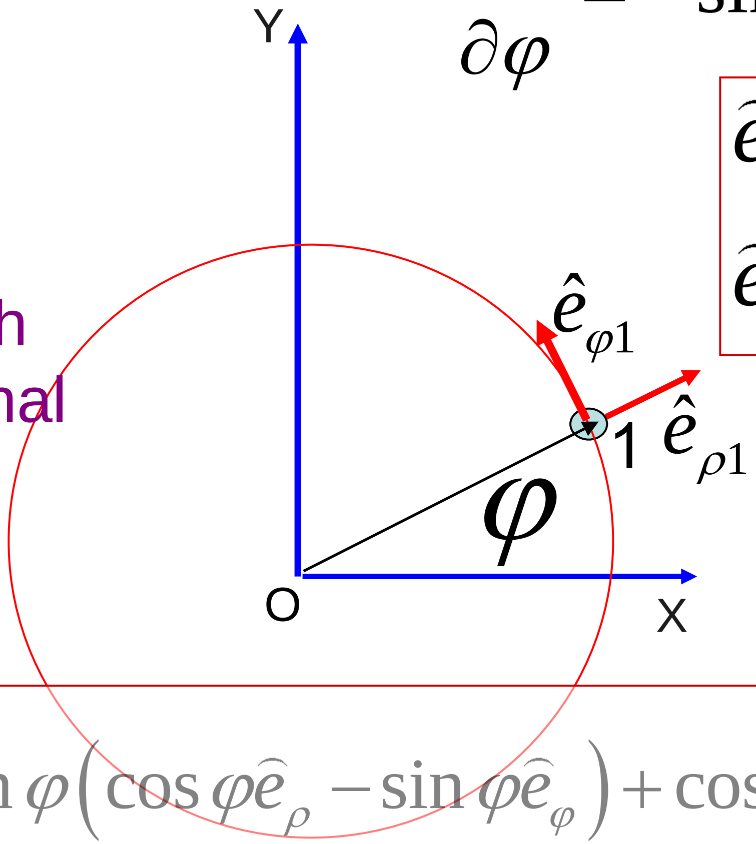
$$\hat{e}_\rho = \hat{e}_\rho(\rho,\varphi)$$

$$\hat{e}_\varphi = \hat{e}_\varphi(\rho,\varphi)$$

$$\hat{e}_\rho = \cos\varphi\,\hat{e}_x + \sin\varphi\,\hat{e}_y$$

$$\frac{\partial \hat{e}_\rho}{\partial \varphi} = -\sin\varphi\,\hat{e}_x + \cos\varphi\,\hat{e}_y$$

How do these unit vectors change with the azimuthal angle?



$$\hat{e}_x = \cos\varphi\,\hat{e}_\rho - \sin\varphi\,\hat{e}_\varphi$$

$$\hat{e}_y = \sin\varphi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\varphi$$

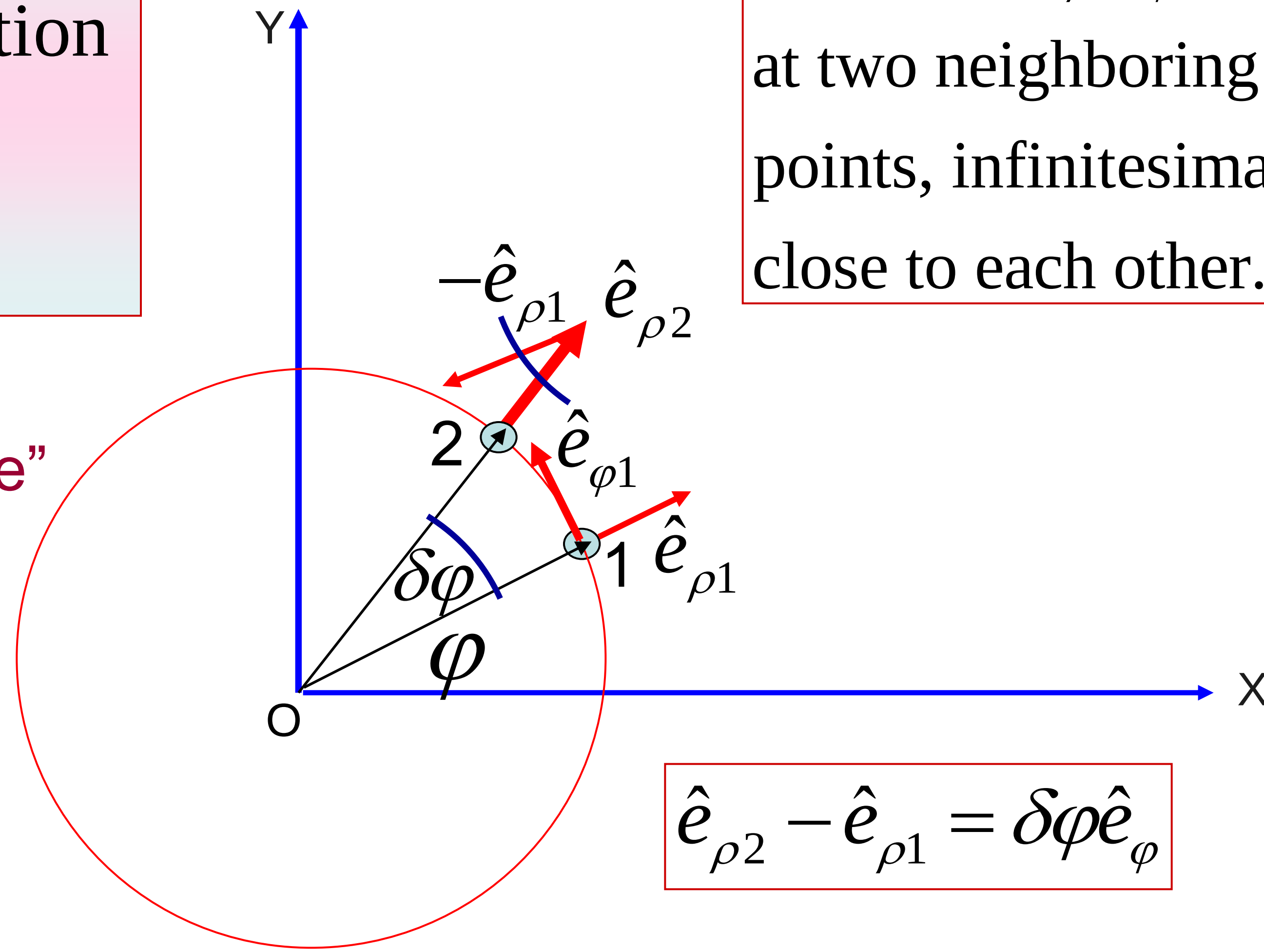
$$\frac{\partial \hat{e}_\rho}{\partial \varphi} = -\sin\varphi\left(\cos\varphi\,\hat{e}_\rho - \sin\varphi\,\hat{e}_\varphi\right) + \cos\varphi\left(\sin\varphi\,\hat{e}_\rho + \cos\varphi\,\hat{e}_\varphi\right)$$

$$= \hat{e}_\varphi$$

Geometrical determination of $\dfrac{\partial \hat{e}_\rho}{\partial \varphi}$

Consider $(\hat{e}_\rho, \hat{e}_\varphi)$ at two neighboring points, infinitesimally close to each other.

“Unit Circle”

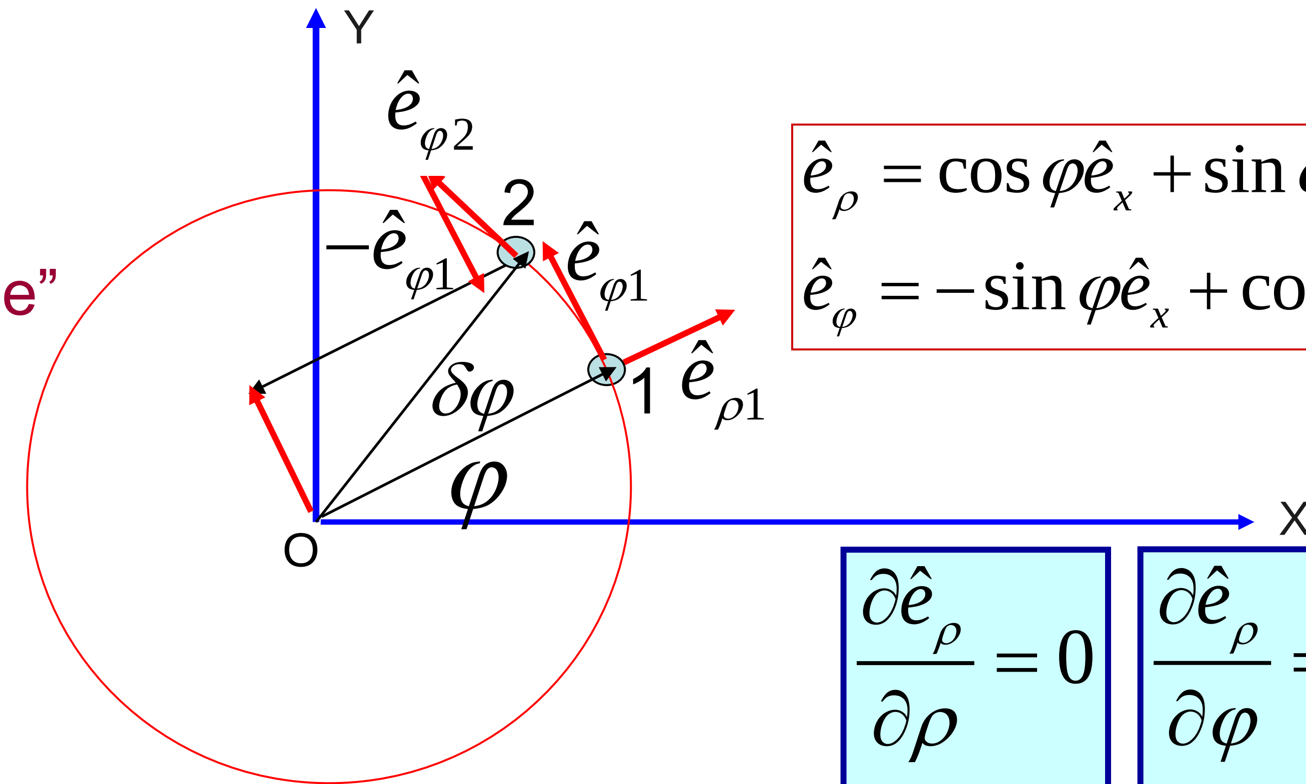


$$\hat{e}_{\rho 2} - \hat{e}_{\rho 1} = \delta\varphi \hat{e}_\varphi$$

$$\lim_{\delta\varphi \to 0} \frac{\hat{e}_{\rho 2} - \hat{e}_{\rho 1}}{\delta\varphi} = \lim_{\delta\varphi \to 0} \frac{\delta \hat{e}_\rho}{\delta \varphi} = \frac{\partial \hat{e}_\rho}{\partial \varphi} = \hat{e}_\varphi$$

$(\hat{e}_\rho, \hat{e}_\varphi)$ are not constant vectors.

$$\hat{e}_\rho = \hat{e}_\rho(\rho, \varphi)$$
$$\hat{e}_\varphi = \hat{e}_\varphi(\rho, \varphi)$$



$$\hat{e}_\rho = \cos\varphi \hat{e}_x + \sin\varphi \hat{e}_y$$
$$\hat{e}_\varphi = -\sin\varphi \hat{e}_x + \cos\varphi \hat{e}_y$$

$$\frac{\partial \hat{e}_\rho}{\partial \rho} = 0 \qquad \frac{\partial \hat{e}_\rho}{\partial \varphi} = \hat{e}_\varphi,$$
$$\frac{\partial \hat{e}_\varphi}{\partial \rho} = 0 \qquad \frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\hat{e}_\rho$$

$$\lim_{\delta\varphi \to 0} \frac{\hat{e}_{\varphi 2} - \hat{e}_{\varphi 1}}{\delta\varphi} = \lim_{\delta\varphi \to 0} \frac{\delta \hat{e}_\varphi}{\delta\varphi} = \frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\hat{e}_\rho$$

chain rule

$$If \;\; \xi = \xi(u) \; and \; u = \phi(x),$$

$$then \; \frac{d\xi}{dx} \; will \; be \; a \; measure \; of \; the$$

$$sensitivity \; of \; \xi \; to \; changes \; in \; x:$$

$$\frac{d\xi}{dx} = \left(\frac{d\xi}{du}\right)\left(\frac{du}{dx}\right)$$

$$If \;\; \xi = \xi(u, v)$$

$$where \; u = u(x), \mathrm{v} = \mathrm{v}(x),$$

$$the \; rate \; at \; which \; \xi \; will$$

$$change \; with \; respect \; to \; x$$

$$will \; be \; given \; by:$$

$$\frac{d\xi}{dx} = \left(\frac{\partial\xi}{\partial u}\right)\left(\frac{du}{dx}\right) + \left(\frac{\partial\xi}{\partial \mathrm{v}}\right)\left(\frac{d\mathrm{v}}{dx}\right)$$

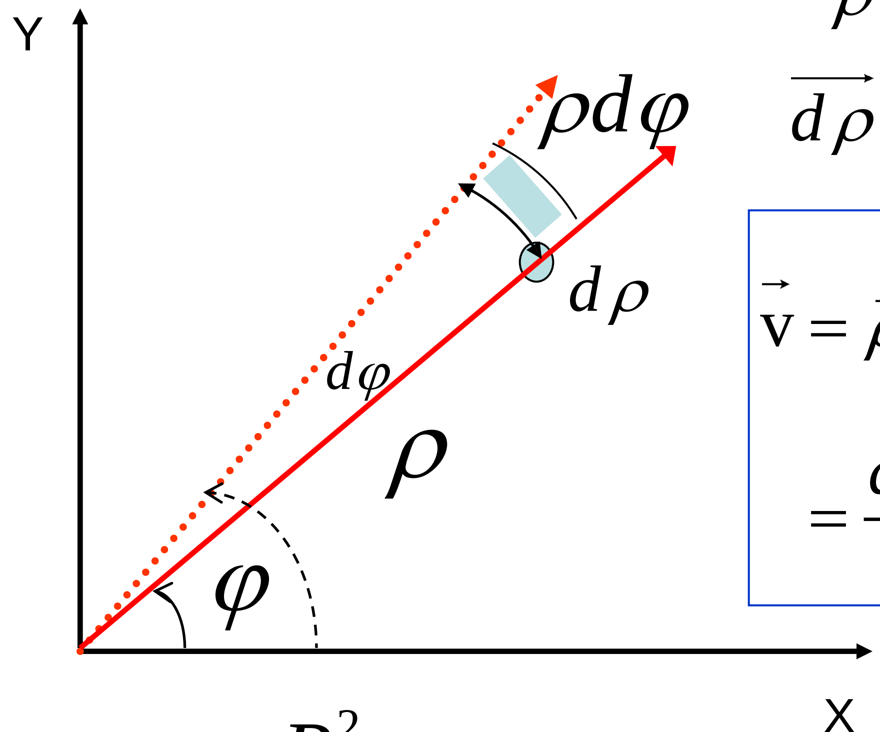
$$If \;\; \xi = \xi(u, \mathrm{v}, x) \; where \; u = u(x), \mathrm{v} = \mathrm{v}(x), \; the \; rate \; at$$

$$which \; \xi \; will \; change \; with \; respect \; to \; x \; will \; be \; given \; by:$$

$$\frac{d\xi}{dx} = \left(\frac{\partial\xi}{\partial u}\right)\left(\frac{du}{dx}\right) + \left(\frac{\partial\xi}{\partial \mathrm{v}}\right)\left(\frac{d\mathrm{v}}{dx}\right) + \left(\frac{\partial\xi}{\partial x}\right).$$

Elemental area in plane polar coordinates

$$dA = \rho d\rho d\varphi$$



$$\int_{\rho=0}^{R}\int_{\varphi=0}^{2\pi}\rho d\rho d\varphi = \frac{R^2}{2}2\pi = \pi R^2$$

Position vector & Velocity in plane polar coordinates

$$\vec{\rho} = \rho\hat{e}_\rho$$

$$\overrightarrow{d\rho} = (d\rho)\hat{e}_\rho + \rho d\hat{e}_\rho$$

$$\vec{v} = \dot{\vec{\rho}} = \frac{d\vec{\rho}}{dt} = \frac{d\left(\rho\hat{e}_\rho\right)}{dt}$$

$$= \frac{d\rho}{dt}\hat{e}_\rho + \rho\frac{d\hat{e}_\rho}{dt}$$

$$\frac{\partial\hat{e}_\rho}{\partial\rho} = 0, \frac{\partial\hat{e}_\rho}{\partial\varphi} = \hat{e}_\varphi,$$

$$\frac{\partial\hat{e}_\varphi}{\partial\rho} = 0, \frac{\partial\hat{e}_\varphi}{\partial\varphi} = -\hat{e}_\rho$$

Motion of a particle in plane polar coordinates

$$\frac{\partial \hat{e}_\rho}{\partial \rho} = 0, \quad \frac{\partial \hat{e}_\rho}{\partial \varphi} = \hat{e}_\varphi,$$

$$\frac{\partial \hat{e}_\varphi}{\partial \rho} = 0, \quad \frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\hat{e}_\rho$$

Time-dependence of unit vectors

$$\frac{d\hat{e}_\rho}{dt} = \frac{\partial \hat{e}_\rho}{\partial \varphi}\dot{\varphi} = \hat{e}_\varphi \dot{\varphi}$$

and

$$\frac{d\hat{e}_\varphi}{dt} = \frac{\partial \hat{e}_\varphi}{\partial \varphi}\dot{\varphi} = -\hat{e}_\rho \dot{\varphi}$$

chain rule

$$\vec{\mathrm{v}} = \dot{\vec{\rho}} = \frac{d\vec{\rho}}{dt} = \frac{d\left(\rho \hat{e}_\rho\right)}{dt} = \frac{d\rho}{dt}\hat{e}_\rho + \rho\frac{d\hat{e}_\rho}{dt}$$

$$= \dot{\rho}\hat{e}_\rho + \rho\dot{\varphi}\hat{e}_\varphi$$

Radial velocity and Azimuthal velocity

$$\vec{\mathrm{v}} = \dot{\rho}\,\hat{\mathrm{e}}_\rho + \rho\dot{\varphi}\ \hat{\mathrm{e}}_\varphi$$ instantaneous velocity

$$\frac{d\hat{e}_\rho}{dt} = \frac{\partial \hat{e}_\rho}{\partial \varphi}\dot{\varphi} = \hat{e}_\varphi \dot{\varphi}$$

and

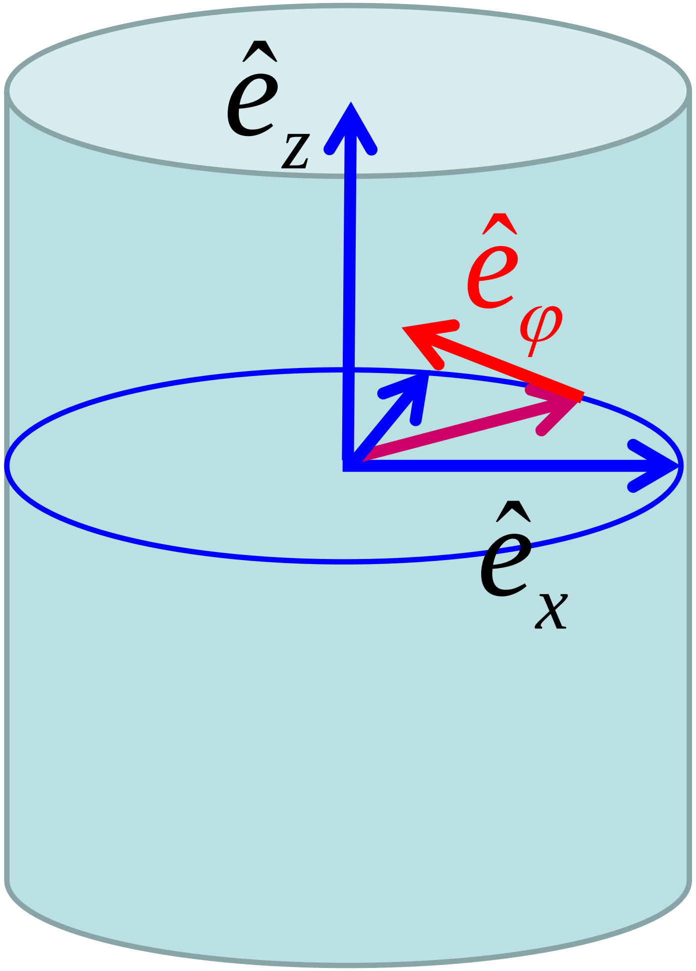
$$\frac{d\hat{e}_\varphi}{dt} = \frac{\partial \hat{e}_\varphi}{\partial \varphi}\dot{\varphi} = -\hat{e}_\rho \dot{\varphi}$$

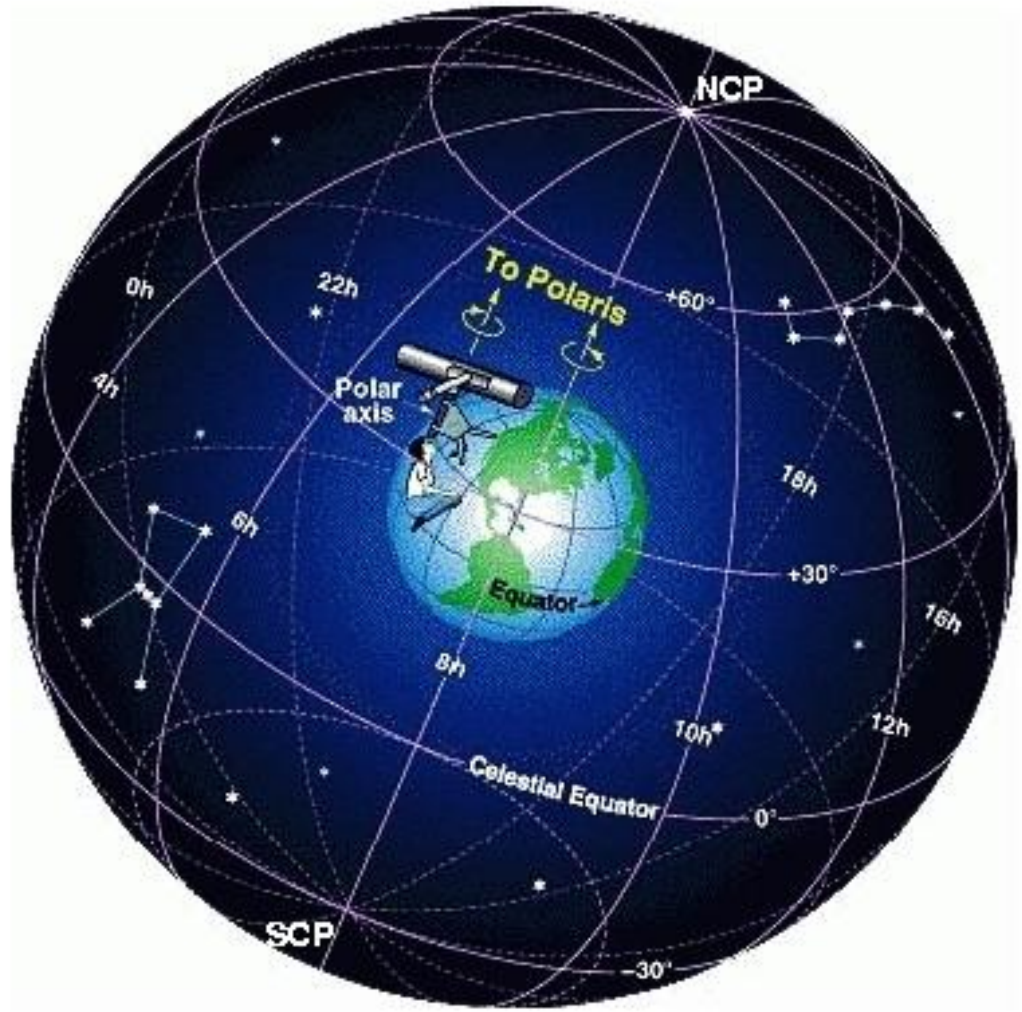
acceleration

$$\vec{a} = \frac{d\vec{\mathrm{v}}}{dt} = \ddot{\rho}\,\hat{\mathrm{e}}_\rho + \dot{\rho}\frac{d\hat{e}_\rho}{dt} + \dot{\rho}\dot{\varphi}\ \hat{e}_\varphi + \rho\ddot{\varphi}\ \hat{e}_\varphi + \rho\dot{\varphi}\frac{d\hat{e}_\varphi}{dt}$$

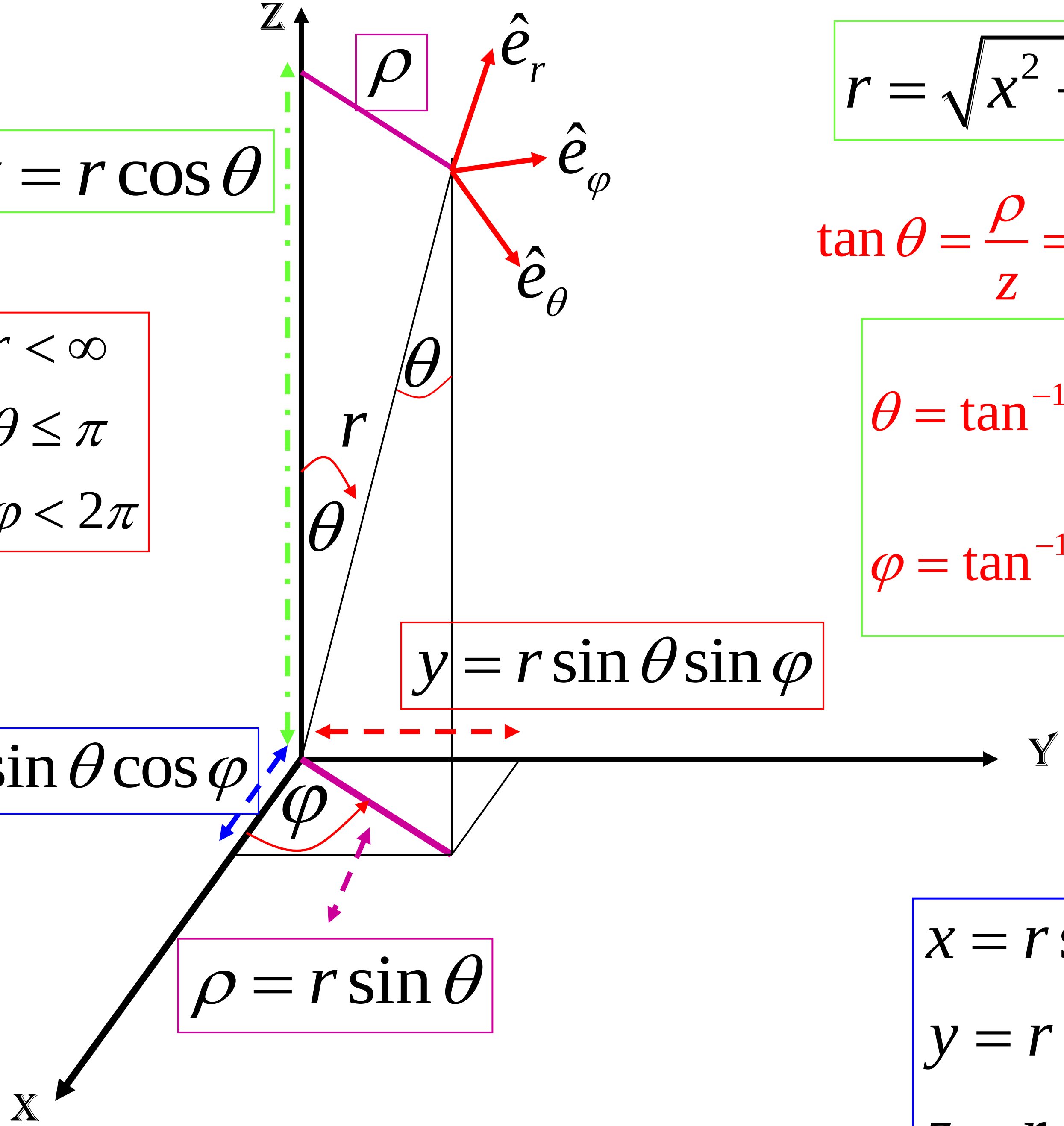
$$\Rightarrow \vec{a} = (\ddot{\rho} - \rho\dot{\varphi}^2)\,\hat{\mathrm{e}}_\rho + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})\hat{\mathrm{e}}_\varphi$$

Cylindrical Polar Coordinates





Spherical Polar Coordinates



$$\rho$$

$$\hat{e}_r \quad \hat{e}_\varphi \quad \hat{e}_\theta$$

$$z = r\cos\theta$$

$$0 \le r < \infty$$
$$0 \le \theta \le \pi$$
$$0 \le \varphi < 2\pi$$

$$y = r\sin\theta\sin\varphi$$

$$x = r\sin\theta\cos\varphi$$

$$\rho = r\sin\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan\theta = \frac{\rho}{z} = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2 + y^2}}{z}$$

$$\varphi = \tan^{-1}\frac{y}{x}$$

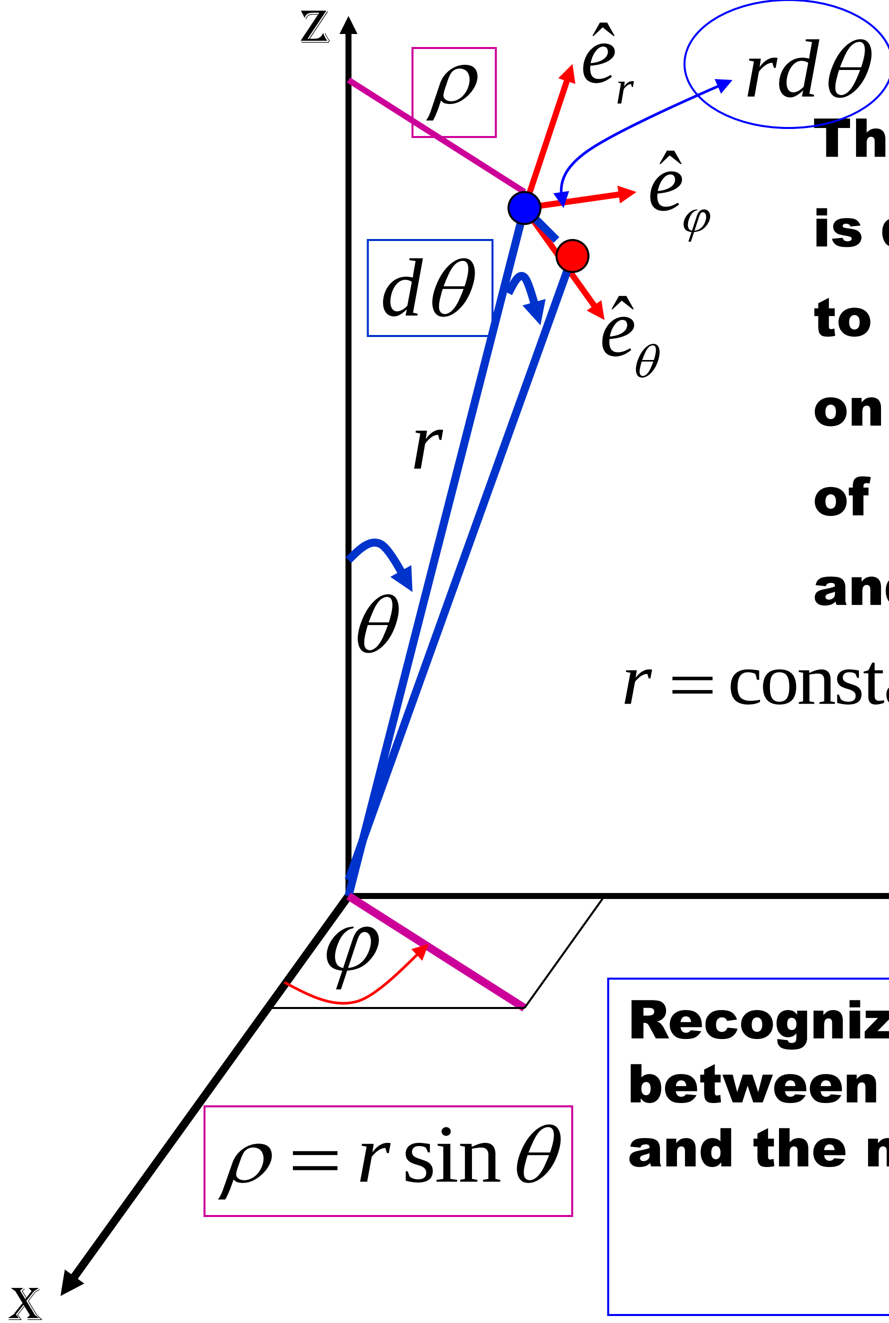
$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

TRANSFORMATIONS OF THE UNIT VECTORS

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\varphi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix}$$

GET THE INVERSE MATRIX, AND WRITE THE INVERSE TRANSFORMATIONS.

$$\begin{bmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\varphi & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\varphi \end{bmatrix}$$



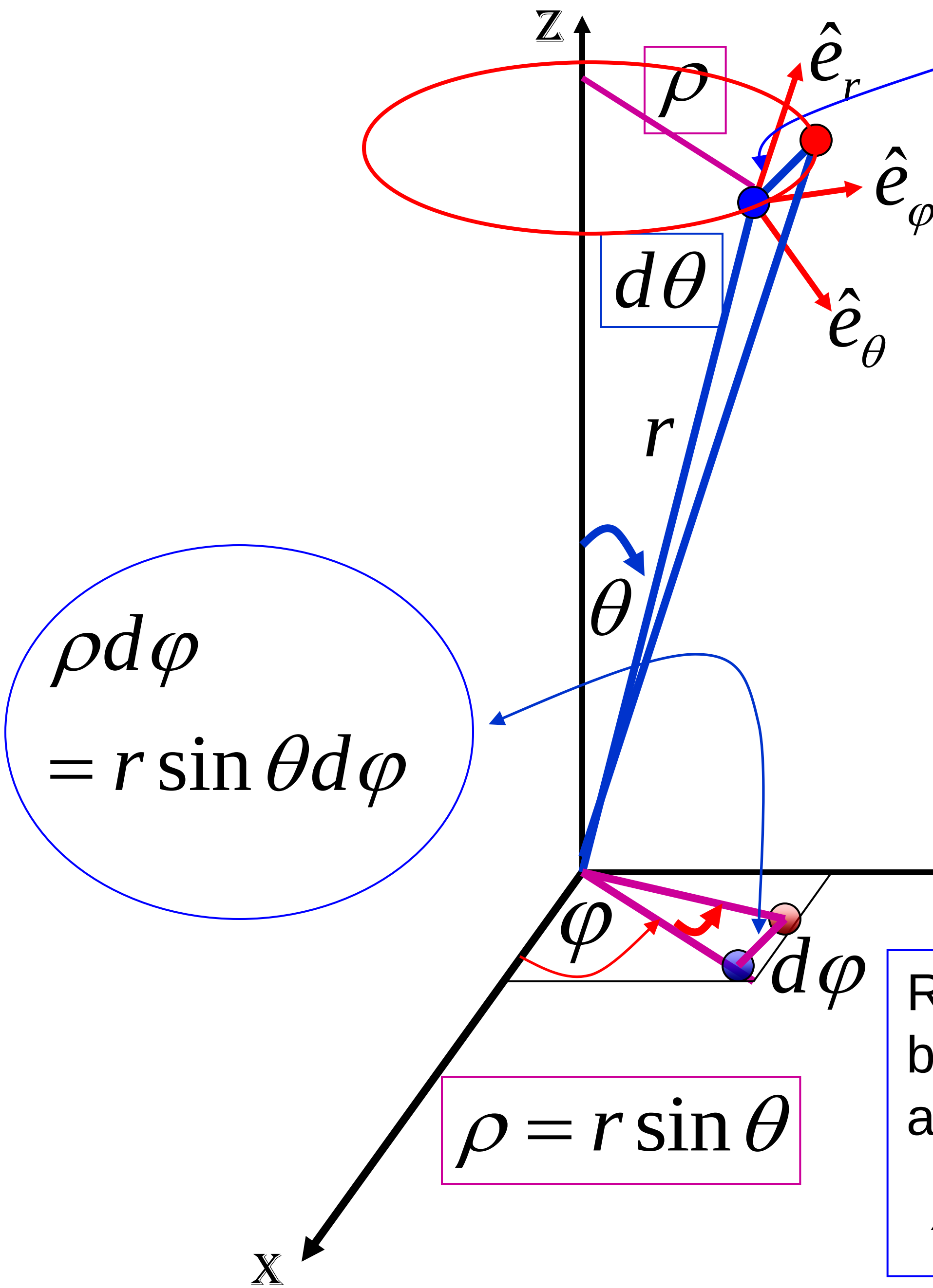
The point (blue dot) **is displaced to a new point** (red dot) **on the sphere of same radius and in the same plane**

$r = \text{constant} \qquad \varphi = \text{constant}$

$\rho = r\sin\theta$

Recognize the distance between the old position and the new position to be

$$rd\theta$$



$$\rho d\varphi = r \sin\theta d\varphi$$

The point (blue dot) is displaced to a new point (red dot) on the sphere $r = \text{constant}$ and on the surface of the inverted cone $\theta = \text{constant}$

Recognize the distance between the old position and the new position to be

$$\rho d\varphi = r \sin\theta d\varphi$$

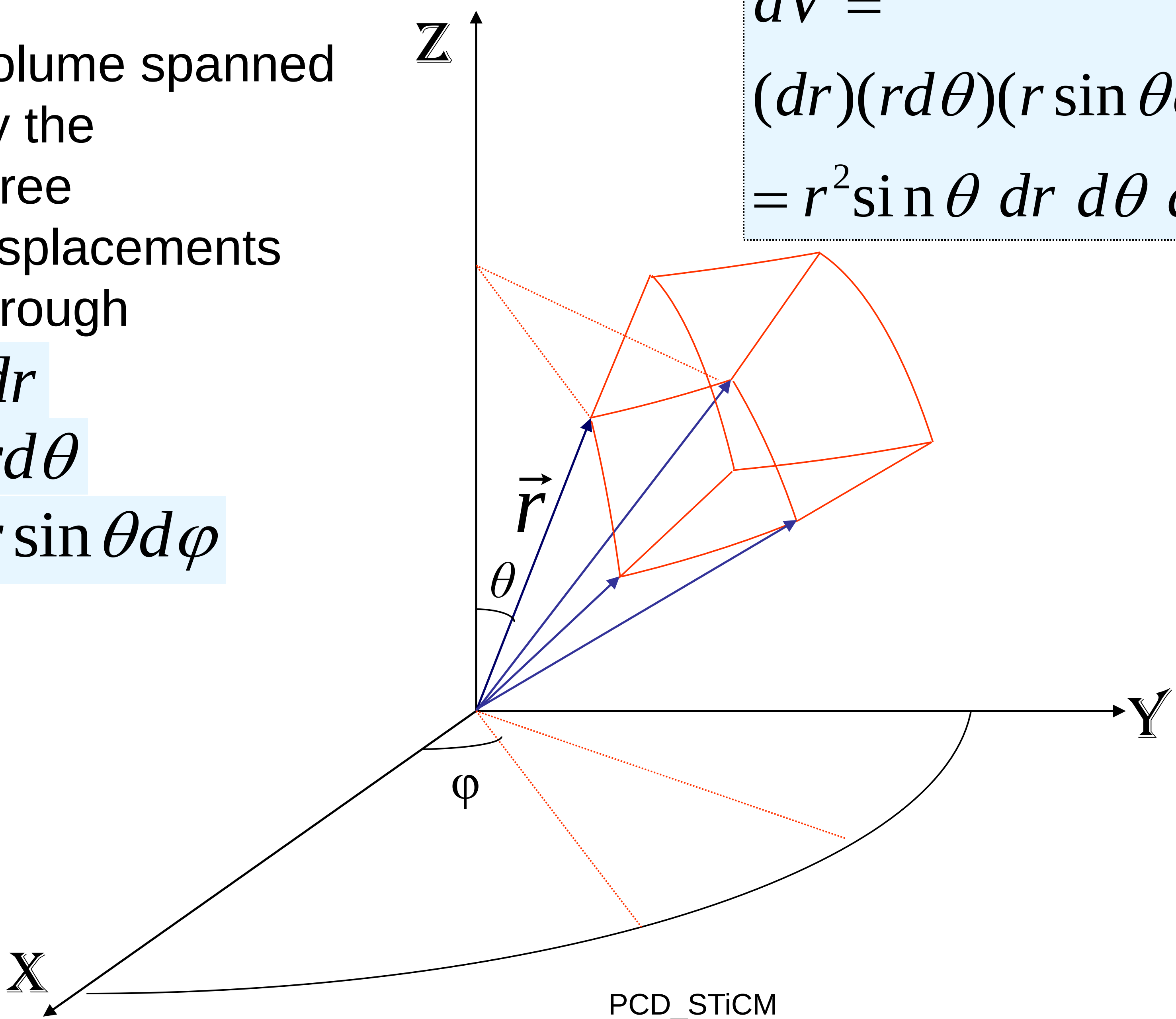
Volume spanned by the three displacements through

dr

$rd\theta$

$r\sin\theta d\varphi$

$$dV = (dr)(rd\theta)(r\sin\theta d\varphi) = r^2\sin\theta\, dr\, d\theta\, d\varphi$$

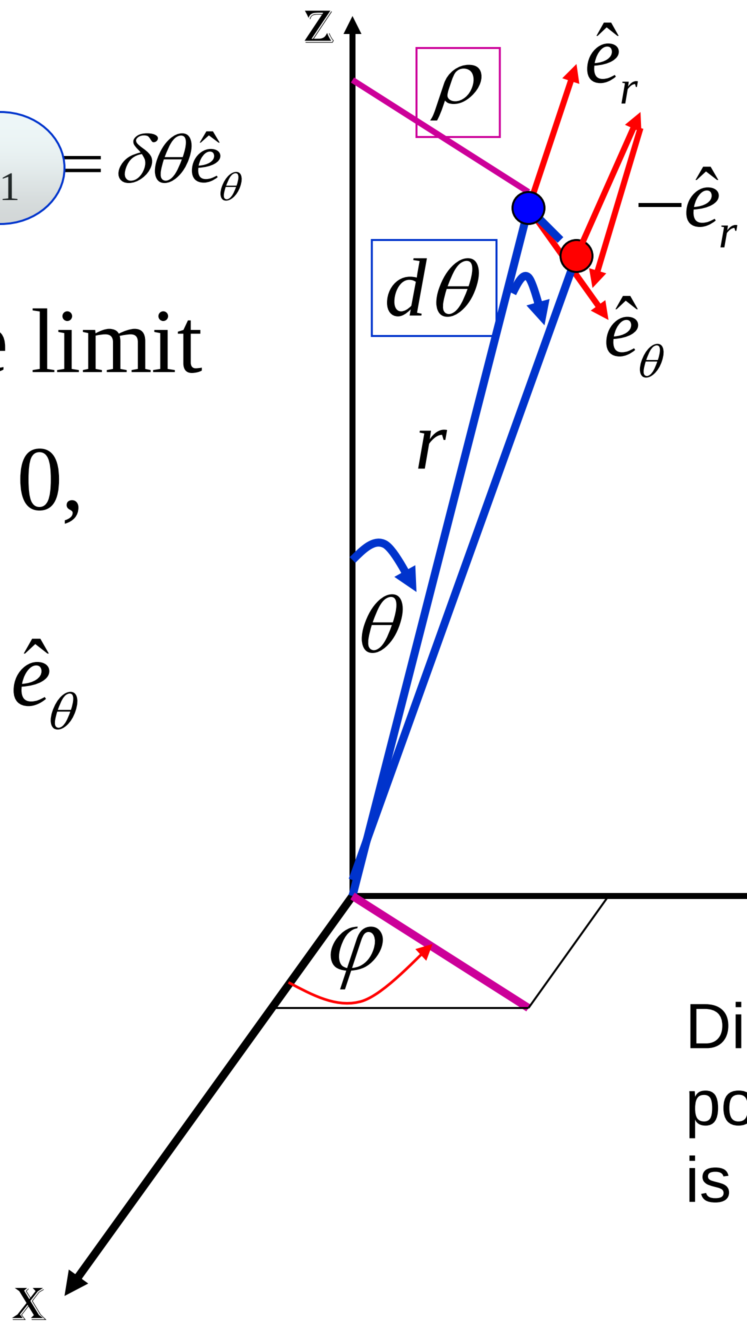


$$\hat{e}_{r2} - \hat{e}_{r1} = \delta\theta\,\hat{e}_\theta$$

In the limit

$\delta\theta \to 0,$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$$



1st position

The point is displaced to a new point on the sphere of same radius

2nd position

$$r = \text{constant}$$

and in the same plane

$$\varphi = \text{constant}$$

Distance between the 1st position and 2nd position is $r\,d\theta$

Partial derivatives of the unit vectors with respect to the coordinates:

$$\frac{\partial \hat{e}_r}{\partial r} = \frac{\partial \hat{e}_\theta}{\partial r} = \frac{\partial \hat{e}_\varphi}{\partial r} = 0$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$$

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$$

$$\frac{\partial \hat{e}_\varphi}{\partial \theta} = 0$$

If imagining complicated geometrical three-dimensional objects is getting difficult, you can use the ‘chain rule’ of taking derivatives to get the partial derivatives of the unit vectors using these transformation rules, as illustrated on the next page.

$$\hat{e}_r = \sin\theta \cos\varphi\, \hat{\mathrm{e}}_x + \sin\theta \sin\varphi\, \hat{\mathrm{e}}_y + \cos\theta\, \hat{\mathrm{e}}_z$$

$$\hat{e}_\theta = \cos\theta \cos\varphi\, \hat{\mathrm{e}}_x + \cos\theta \sin\varphi\, \hat{\mathrm{e}}_y - \sin\theta\, \hat{\mathrm{e}}_z$$

$$\hat{e}_\varphi = -\sin\varphi\, \hat{\mathrm{e}}_x + \cos\varphi\, \hat{\mathrm{e}}_y$$

Use of ‘chain rule’ to get the partial derivatives of the unit vectors using the transformation rules for the unit vectors.

$$\hat{e}_r = \sin\theta\cos\varphi\,\hat{\mathrm{e}}_x + \sin\theta\sin\varphi\,\hat{\mathrm{e}}_y + \cos\theta\,\hat{\mathrm{e}}_z$$

$$\hat{e}_\theta = \cos\theta\cos\varphi\,\hat{\mathrm{e}}_x + \cos\theta\sin\varphi\,\hat{\mathrm{e}}_y - \sin\theta\,\hat{\mathrm{e}}_z$$

$$\hat{e}_\varphi = -\sin\varphi\,\hat{\mathrm{e}}_x + \cos\varphi\,\hat{\mathrm{e}}_y$$

For example:

$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\cos\varphi\,\hat{e}_x - \sin\varphi\,\hat{e}_y$$

$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\cos\varphi\left(\sin\theta\cos\varphi\,\hat{e}_r + \cos\theta\cos\varphi\,\hat{e}_\theta - \sin\varphi\,\hat{e}_\varphi\right) - \sin\varphi\left(\sin\theta\sin\varphi\,\hat{e}_r + \cos\theta\sin\varphi\,\hat{e}_\theta + \cos\varphi\,\hat{e}_\varphi\right)$$

$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\sin\theta\,\hat{e}_r - \cos\theta\,\hat{e}_\theta$$

Other partial derivatives can be obtained equally easily, and left for your to do as an exercise!

$$\frac{\partial \hat{e}_r}{\partial r} = \vec{0}$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$$

$$\frac{\partial \hat{e}_r}{\partial \varphi} = \sin\theta \hat{e}_\varphi$$

$$\frac{\partial \hat{e}_\theta}{\partial r} = \vec{0}$$

$$\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$$

$$\frac{\partial \hat{e}_\theta}{\partial \varphi} = \cos\theta \hat{e}_\varphi$$

$$\frac{\partial \hat{e}_\varphi}{\partial r} = \vec{0}$$

$$\frac{\partial \hat{e}_\varphi}{\partial \theta} = \vec{0}$$

$$\frac{\partial \hat{e}_\varphi}{\partial \varphi} = -\cos\theta \hat{e}_\theta - \sin\theta \hat{e}_r$$

MOTION IN IN SPHERICAL POLAR:

VELOCITY AND ACCELERATION

Infinitesimal displacement

Position vector $\vec{r} = r\hat{e}_r$

$$d\vec{r} = dr\hat{e}_r + rd\hat{e}_r$$

$$d\vec{r} = dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\varphi\hat{e}_\varphi$$

$$\Rightarrow d\vec{r} = dr\hat{e}_r + r\frac{\partial\hat{e}_r}{\partial\theta}\delta\theta + r\frac{\partial\hat{e}_r}{\partial\varphi}\delta\varphi$$

Motion in in spherical polar:
Velocity and acceleration

$$d\vec{r} = dr\hat{e}_r + rd\theta\hat{e}_\theta + r\sin\theta d\varphi\hat{e}_\varphi$$

$$\vec{\mathrm{v}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\dot{\varphi}\hat{e}_\varphi$$

$$\vec{a} = \frac{d\vec{\mathrm{v}}}{dt} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\varphi}^2)\hat{e}_r$$

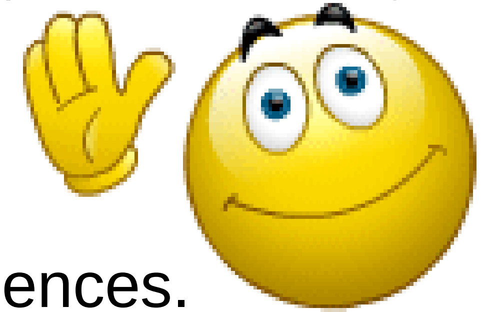
$$+(2\dot{r}\dot{\theta} - r\sin\theta\cos\theta\dot{\varphi}^2 + r\ddot{\theta})\hat{e}_\theta$$

$$+(2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\varphi}\dot{\theta}\cos\theta + r\sin\theta\ddot{\varphi})\hat{e}_\varphi$$

General Reference on Vector analysis :

[1] Berkeley Physics Course, Vol.1. ‘Mechanics’
[2] Davis: ‘Classical Mechanics’

SUPPLEMENTARY OPTIONAL READING:
General Reference on Astronomy :
Patrik Moore: International Encyclopedia of Astronomy.
Carl Sagan: Cosmos



Slightly advanced references:

Arfken: Mathematical Methods for Physicists.
Boas: Mathematical methods in Physical Sciences.

WE WILL TAKE A BREAK…

…… ANY QUESTIONS ?

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Next, Unit 4: Dynamical Symmetry of the Kepler Problem